MATHEMATICAL MODEL OF EUROPEAN OPTION PRICING IN INCOMPLETE MARKET WITHOUT TRANSACTION COSTS (DISCRETE TIME). PART I.

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For European option in multidimensional incomplete market without transaction costs we design discreet time pricing model. At first the following auxiliary problem is to be considered: to find the upper guaranteed value for the expected risk depending exponentialy on a shortage. The upper guaranteed value is a minimax of the expected risk. First we take supremum over a set of equivalent probability measures. Then we take infimum over a set of self-financing portfolios. Here we find conditions for the existence of a portfolio such that an infimum is attained. We use this result to find a generalized optional decomposition for a contingent claim. Further, we obtain conditions for the existence of a probability measure such that the expected risk is maximal with respect to the measure. This measure turned out to be martingale and discreet and it does not belong to the set of equivalent measures. Finally, we demonstrate that our auxiliary results make it possible to obtain explicit pricing formulas for an European option in an incomplete market without transaction costs. In part II of the paper we present example models of European options' pricing in a one-dimensional market and in a market, where support of basic probability measure is compact.
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Introduction

1. For European option in multidimensional incomplete market without transaction costs we design discreet time pricing model. The core problem here is to choose a probability measure with respect to which one should value an option. In complete markets it is a rule to value an option with respect to the unique equivalent martingale measure. In incomplete markets there is a continuum set of such a measures.

The authors of [5], [8], [9], [10], [12], [17], [19] suggest choosing an equivalent martingale probability measure such that the price of an European option is maximal. In these papers, one can find methods for constructing a portfolio and calculating the price of an option for different models of incomplete markets. The methods are based on optional decomposition of supermartingales.

The authors of [5] establish an optional decomposition under assumption that the evolution of risky assets' prices is represented by a diffusion process with jumps. They prove the existence of a superhedging portfolio with consumption and find the price of an option using this decomposition.

The authors of [9], [12], [19] prove the existence of an optional decomposition (for contingent claims) with respect to a class of equivalent martingale measures in incomplete arbitrage-free markets without transaction costs. In these papers, they assume that the evolution of risky assets' prices is given by a semimartingale. In that case, they provide a method for calculating the European options' prices in incomplete markets in terms of this decomposition.

In [8], [10], [17], the authors also establish the existence of an optional decomposition of contingent claims with respect to equivalent martingale measures in incomplete arbitrage-free markets without transaction costs in discrete time and provide a method for option pricing in terms of this decomposition.

In [11] for an abstract model of market they have found necessary and sufficient conditions for existence of representation for an upper hedging price $\pi(B)$, where B is a nonnegative contingent claim in the form of functional $\sup\limits_{\eta\in D} E\eta B$ and D is a set of nonnegative random variables. There is also a detailed review of results on theory of European options' superhedging in incomplete markets.

Note that this approach to option pricing requires calculation of a essential supremum over a set of martingale measures of functionals' conditional expectations, where the functionals are defined on the trajectories of risky assets' prices; calculation of this essential supremum is a substantial mathematical problem. For this reason, in [5], [8], [9], [10], [12], [17], [19], there is a lack of explicit formulas describing the portfolio process and the corresponding process of the capital evolution. It is well known [1], [6], [13] that the calculation of essential supremum of additive or multiplicative functional's conditional expectation (where a functional is defined on trajectories of a controlled random process) is an object of stochastic optimal control. In this theory, they solve the problem using methods of the stochastic dynamic programming (see, e.g., [5]).

2. In this paper, we design pricing model for European option in incomplete market without transaction costs when time is discrete applying the minimax principle (in contrast to [5], [8], [9], [10], [12], [17], [19]) that can be formulated as follows: (i) as far as the probability distribution of the

risky assets' price evolution process is unknown, one should suppose that it maximizes the price of an European option; (ii) one should buy with minimal capital as many risky assets as to be sure to cover an option's contingent claim. In this paper, the realization of this principle is based on the following two opportunities. The first one is the reduction of a minimax calculation problem to a game problem of optimal stochastic control. The second one is based on the reduction of an European options pricing problem to a game problem of optimal stochastic control with a multiplicative functional. The last opportunity follows from the results of [4].

3. Let us outline our approach to European option's price modelling. We consider a multidimensional incomplete market specified by a semimartingale and a European option with finite time horizon and bounded pay-off.

At first we study auxiliary game problem. Specifically, suppose that there are two players watching the d-dimensional sequence of risky assets' prices. The first player represents a market. Its strategies are probability measures defined on the trajectories of risky assets' prices and equivalent to some basic measure. The second player manages assets. His strategies are self-financing portfolios (described by multidimensional predictable sequences). We suppose that the risk function (the payoff function of the second player): (i) depends on his shortfall; (ii) is exponential (this choice will be explained later). As in [10], the shortfall is the difference between a contingent claim and the profit gained by the second player from the portfolio during the option lifetime, i.e., We also suppose that the players are "rational" and choose their strategies independently. The first player maximizes the expected risk over a set equivalent probability measures. The second player minimizes the expected risk over admissible (in a sense clarified below) self-financing portfolios. Therefore we have the minimax problem.

The idea to consider such a problem goes back to [4], where the problem was solved for a special case. The authors used the method of stochastic dynamic programming to prove the existence of the S-representation of martingales (for the definition of S-representation, see [17]). In this paper, we generalize this result (see Theorem 4). Note that we have chosen the exponential risk function just because it allows us to apply the above-stated method. The solution of our auxiliary problem (2) allows us: (1) to establish an analogue of the optional decomposition for any $\mathcal{F}^S$-measurable bounded function f_N, i.e., for any contingent claim in the European options' problem in incomplete markets without transaction costs; (2) to investigate the properties of the measure with respect to which the essential supremum of Lebesgue integral is attained; (3) to choose a probability measure with respect to which one should estimate an option. Finally, all these made it possible to design our pricing model for European option in incomplete market, namely:

(1) to find a portfolio of assets at any moment and the corresponding capital;
(2) to calculate the upper bound for spread.

4. Let us briefly discuss the structure of the paper. The paper is in two parts. Here contents of the first part is outlined. Section 1 deals with our auxiliary game problem (2). First, it is shown that we can use the method of stochastic dynamic programming for that problem, i.e. we prove that sequence of upper guaranteed values satisfies recurrent relation (5) (Theorem 1). Further, we determine conditions for the existence of admissible portfolio such that the outer essential infimum is attained (Theorem 3). We use this result to prove that the contingent claim allows an optional decomposition with respect to the class of equivalent measures (Theorem 4). Further, we find conditions for the existence of a probability measure with respect to which the inner essential supremum is attained (Theorem 6). From these results existence condition for solution of auxiliary problem follows (Theorem 8). For convenience of reading all proves are grouped in Section 3.

In Section 2, we design our pricing model for European option in an incomplete market without transaction costs when time is discrete. First, we use Theorem 4 (an analog of the optional decomposition) to link auxiliary problem (2) and the superhedging problem [17]; namely, we construct perfect superhedging portfolio for a European option in an incomplete market without transaction costs (Theorem 10). Also here we prove, that the capital of the above-mentioned perfect superhedging portfolio (constructed for the exponential risk function) is less than or equal to the capital of any other perfect superhedging portfolio at any time moment. This means that the capital of the minimal prefect superhedging portfolio coincides with the upper bound of the spread. Further, we prove that the measure with respect to which essential supremum is attained (constructed in Section 1, we call it the worst-case measure) is a martingale one (Theorem 11). So, a contingent claim admits an S-representation [17] with respect to the measure (Theorem 12). Besides, we prove that there is discrete worst-case measure (Theorem 13) and in the case of incomplete market it is not equivalent to the basic measure (Remark 7). It follows from these statements that we can identify the initial incomplete market with a complete one with respect to the worst-case measure and the corresponding minimal perfect superhedging portfolio has zero consumption. This portfolio is called a minimax hedging portfolio. Note that the capital of the minimax hedgeing portfolio coincides with the upper bound of the spread. And as the market is complete with respect to the worst-case measure, it is possible to calculate it explicitly. All statements of Section 2 are proved in Section 4.

The second part of the paper consists of two sections with examples. Using our pricing model of part I, in Section 5 we construct the minimax hedging portfolio for a European option in a one-dimensional finite incomplete market. In Section 6, we give an example of a European option' pricing in an one-dimensional incomplete compact market.

§1. Auxiliary minimax problem

In this section we consider auxiliary minimax problem (2) and, as a result, find existence conditions for it's solution. These results are essential for our prising model to be constructed in Section 2. Though problem (2) and our approach to the solution are interesting in themselves.

1.1. First let us introduce some notation.

1.1.1. Let $\{S_t, \mathcal{F}_t\}_{t\in\mathbb{N}^+}$ be a d-dimensional adapted random sequence on the stochastic basis $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{N}^+}, \mathsf{P}\right)$. Suppose that:

(i) a probability measure P is fixed (this measure is said to be *basic* [17]);

(ii) for any $t \in \mathbb{N}^+$ the σ-algebra $\mathcal{F}_t = \mathcal{F}_t^S \triangleq \sigma(S_u, u \le t)$.

Together stochastic basis $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{N}^+}, \mathsf{P}\right)$ and $\{S_t, \mathcal{F}_t\}_{t\in\mathbb{N}^+}$ specify a financial $\{1, S\}$-market [17].

By $\Re_N$ we denote the set of all probability measures on $\left(\Omega, \mathcal{F}, \left(\mathcal{F}_t^S\right)_{t\in N_0}\right)$ such that any measure $\mathsf{Q} \in \Re_N$ is equivalent to the basic measure P. Without loss of generality we suppose that $\mathsf{P} \in \Re_N$; so, $\Re_N \neq \varnothing$. The set of all martingale measures (i.e. measures with respect to which $\{S_t, \mathcal{F}_t\}_{t\in N_0}$ is a local martingale, see [17]) is denoted by $\mathfrak{M}_N$.

The expectation of a random variable θ with respect to a probability measure Q (P) is denoted by $\mathsf{E}^{\mathsf{Q}}\theta$ $\left(\mathsf{E}^{\mathsf{P}}\theta\right)$, and $\mathsf{E}^{\mathsf{Q}}\left(\theta|\mathcal{F}_t^S\right)$ is the conditional expectation with respect to the measure Q and the σ-algebra $\mathcal{F}_t^S$.

1.1.2. Let $f_N(S_\bullet)$ be a bounded $\mathcal{F}_N^S$-measurable random variable, where $N \in \mathbb{N}^+$. Here $f_N(S_\bullet)$ (or short f_N) represents pay-off function of European option with *horizon* N [17], [10]. We write $N_k \triangleq \{k, k+1, k+2, \ldots, N\}$, $k \in \{0, ..., N\}$.

A d-dimensional $\mathcal{F}^S$-predictable sequence is called a *strategy* and is denoted by $\gamma_1^N \triangleq \{\gamma_t\}_{t\in N_1}$, where $N_1 \triangleq \{1, 2, \ldots, N\}$. The vector γ_t is a control at a time $t \in N_1$. By U_1^N we denote the set of strategies. Let $\widetilde{U}_1^N$ be an arbitrary subset of U_1^N. By $\widetilde{U}_{t_1}^{t_2}$ we denote the reduction of the set $\widetilde{U}_1^N$ to $\{t_1, \ldots, t_2\} \subseteq N_1$, where $t_1, t_2 \in N_1$ and $t_2 \ge t_1$. Thus, we will use the following notation $\gamma_{t_1}^{t_2} \in \widetilde{U}_{t_1}^{t_2}$, where $\gamma_{t_1}^{t_2} \triangleq \{\gamma_{t_1}, \ldots, \gamma_{t_2}\}$.

1.1.3.

Definition 1 *A pair* $\left(\mathsf{Q}, \gamma_{t+1}^N\right) \in \Re_N \times U_{t+1}^N$ *is called a t-bistrategy,* $t \in N_1$*;* $\left(\mathsf{Q}, \gamma_1^N\right) \in \Re_N \times U_1^N$ *is a bistrategy, and* $\gamma_{t+1}^N \in \widetilde{U}_{t+1}^N$ *is a t-strategy.*

Definition 2 *An estimate of a t-bistrategy* $\left(\mathsf{Q}, \gamma_{t+1}^N\right)$*,* $t \in N_1$*, is an* $\mathcal{F}_t^S$*-measurable random variable (denoted by* $I_t^{\mathsf{Q},\gamma_{t+1}^N}\left(S_0^t\right)$*) defined by*

$$I_t^{\mathsf{Q},\gamma_{t+1}^N}\left(S_0^t\right) \triangleq \mathsf{E}^{\mathsf{Q}}\left[\exp\left\{f_N\left(S_\bullet\right) - \sum_{i=t+1}^{N}\left(\gamma_i, \Delta S_i\right)\right\}|\mathcal{F}_t^S\right]. \qquad (1)$$

Above $(\bullet, \bullet)$ is the scalar product in a multidimensional Euclidean space, $\Delta S_i \triangleq S_i - S_{i-1}$.

Definition 3 *A random variable* $f_N\left(S_\bullet\right)$ *and a strategy* γ_1^N *are admissible if* $\operatorname*{ess\,sup}_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}} I_0^{\mathsf{Q},\gamma_1^N}\left(S_0\right) < \infty \quad \mathsf{Q} - a.s.$

As f_N is bounded Q-a.s., so pair $\left(f_N, \gamma_1^N\right)$ is admissible if $\operatorname*{ess\,sup}_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}} \exp\left\{-\sum_{i=1}^{N}\left(\gamma_i, \Delta S_i\right)\right\} < \infty \quad \mathsf{Q} - a.s.$ For given f_N, by D_1^N we denote the set of all admissible strategies γ_1^N. Note, that $D_1^N \neq \varnothing$ as trivial strategy belongs to admissible pair $\left(f_N, \gamma_1^N\right)$ for any Q-a.s. bounded f_N.

Definition 4 *A bistrategy* $\left(\mathsf{Q}, \gamma_1^N\right) \in \Re_N \times D_1^N$ *is said to be admissible.*

We consider the following problem:

$$I_0^{\mathsf{Q},\gamma_1^N}\left(S_0\right) \longrightarrow \operatorname*{ess\,inf}_{\gamma_1^N \in D_1^N} \operatorname*{ess\,sup}_{\mathsf{Q}\in\Re_N}. \qquad (2)$$

Definition 5 *The random variable* $\overline{V}_0 \triangleq \operatorname*{ess\,inf}_{\gamma_1^N \in D_1^N} \operatorname*{ess\,sup}_{\mathsf{Q}\in\Re_N} I_0^{\mathsf{Q},\gamma_1^N}\left(S_0\right)$ *is called the upper guaranteed value.*

Definitions of ess inf and ess sup with respect to a basic measure P can be found in [7], [10], [17], [18].

Note that $\overline{V}_0$ is an $\mathcal{F}_0^S$-measurable random variable.

Definition 6 *The triplet* $\left(\mathsf{Q}^*, \gamma_1^{*N}, \overline{V}_0\right)$*:*

$$\overline{V}_0 = I_0^{\mathsf{Q}^*,\gamma_1^{*N}}\left(S_0\right). \qquad (3)$$

is a solution of the minimax problem (2); here the probability measure Q^* *is called the worst-case measure, the strategy* $\gamma_1^{*N} \in D_1^N$ *is called the minimax strategy and together* $\left(\mathsf{Q}^*, \gamma_1^{*N}\right)$ *are referred to as the minimax bistrategy.*

1.2. To solve problem (2) we use the stochastic version of dynamic programming. So we define sequence of upper guaranteed values as follows.

Definition 7 *A random variable*

$$\overline{V}_t \triangleq \operatorname*{ess\,inf}_{\gamma_{t+1}^N \in D_{t+1}^N} \operatorname*{ess\,sup}_{\mathsf{Q} \in \Re_N} I_t^{\mathsf{Q},\gamma_{t+1}^N}\left(S_0^t\right) \tag{4}$$

is called the upper guaranteed value at a time $t \in N_0$.

According to the definitions of ess inf and ess sup (see [10], [17]) $\overline{V}_t$ is an $\mathcal{F}_t^S$-measurable random variable.

In this section, we give a recurrent relation for the sequence $\left\{\overline{V}_t, \mathcal{F}_t^S\right\}_{t\in N_0}$.

Theorem 1 *Suppose* $f_N(S_\bullet)$ *is an* $\mathcal{F}_N^S$*-measurable bounded random variable. Then* $\left\{\overline{V}_t, \mathcal{F}_t^S\right\}_{t\in N_0}$ *satisfies the recurrent relation* P*-a.s.*

$$\begin{cases} \overline{V}_t = \operatorname*{ess\,inf}_{\gamma \in D_{t+1}} \operatorname*{ess\,sup}_{\mathsf{Q} \in \Re_N} \mathsf{E}^{\mathsf{Q}}\left[\overline{V}_{t+1} e^{-(\gamma,\Delta S_{t+1})} | \mathcal{F}_t^S\right], \quad 0 \le t < N, \\ \overline{V}_t|_{t=N} = e^{f_N(S_\bullet)}. \end{cases} \tag{5}$$

Corollary 1 *Suppose, the assumptions of Theorem 1 are satisfied. Then*

(1) for any $t \in N_1$ *and* $\mathsf{Q} \in \Re_N$*, the following inequality holds* P*-a.s.:*

$$\overline{V}_{t-1} \ge \operatorname*{ess\,inf}_{\gamma \in D_t} \mathsf{E}^{\mathsf{Q}}\left[\overline{V}_t e^{-(\gamma,\Delta S_t)} | \mathcal{F}_{t-1}^S\right]; \tag{6}$$

(2) for any $t \in N_1$ *and* $\gamma \in D_t$*, the following inequality holds* P*-a.s.:*

$$\overline{V}_{t-1} \le \operatorname*{ess\,sup}_{\mathsf{Q} \in \Re_N} \mathsf{E}^{\mathsf{Q}}\left[\overline{V}_t e^{-(\gamma,\Delta S_t)} | \mathcal{F}_{t-1}^S\right]. \tag{7}$$

1.3. Upper guaranteed value might be a priori estimated as follows.

Theorem 2 *Suppose:*

(1) conditions of Theorem 1 are satisfied;

(2) there exists constant c_2 *such that* $|f_N(S_\bullet)| \le c_2$ P*-a.s.;*

(3) $\Re_N \cap \mathfrak{M}_N \neq \varnothing$.

Than for any $t \in N_1$ *the following inequalities hold* P*-a.s.*

$$e^{-c_2} \le \overline{V}_t \le e^{c_2}. \tag{8}$$

1.4. In this subsection we give a sufficient condition for the "outer" essential infimum in (5) to be attained.

Theorem 3 *Suppose:*

(1) the assumptions of Theorem 1 are satisfied;

(2) $\Re_N \cap \mathfrak{M}_N \neq \varnothing$.

Then there is a strategy $\{\gamma_t^*\}_{t \in N_1} \in D_1^N$ such that for any $t \in N_1$, P-a.s.

$$\overline{V}_t = \operatorname*{ess\,inf}_{\gamma \in D_{t+1}} \operatorname*{ess\,sup}_{\mathsf{Q} \in \Re_N} \mathsf{E}^{\mathsf{Q}} \left[\overline{V}_{t+1} e^{-(\gamma, \Delta S_{t+1})} | \mathcal{F}_t^S \right] = \tag{9}$$

$$= \operatorname*{ess\,sup}_{\mathsf{Q} \in \Re_N} \mathsf{E}^{\mathsf{Q}} \left[\overline{V}_{t+1} e^{-\left(\gamma_{t+1}^*, \Delta S_{t+1}\right)} | \mathcal{F}_t^S \right].$$

Moreover, for any $t \in N_1$ and $\mathsf{Q} \in \Re_N$ the following inequality is true:

$$\overline{V}_{t-1} \geq \mathsf{E}^{\mathsf{Q}} \left[\overline{V}_t e^{-(\gamma_t^*, \Delta S_t)} | \mathcal{F}_{t-1}^S \right] \quad \mathsf{P}-a.s. \tag{10}$$

Remark 1 *It follows from Corollary 1 that for any $t \in N_0$ and $\mathsf{Q} \in \Re_N$,*

$$\overline{V}_t \geq I_t^{\mathsf{Q}, \gamma_{t+1}^{*N}} \left(S_0^t \right) \quad \mathsf{Q}\left(\mathsf{P}\right) - a.s.,$$

where $\gamma_{t+1}^{*N} \in D_{t+1}^N$ is defined by (9).

1.5. In this subsection we use Theorem 3 to obtain a condition for any $\mathcal{F}_N^S$-measurable bounded random variable to have a decomposition similar to the optional decomposition [10], [17].

Theorem 4 *Let $\left\{ \overline{V}_t, \mathcal{F}_t^S \right\}_{t \in N_0}$ be defined by (5). Suppose, there exist a strategy $\{\gamma_t^*\}_{t \in N_1} \in D_1^N$ satisfying (9) for any $t \in N_1$. Then for any $t \in N_1$ and $\mathsf{Q} \in \Re_N$, the sequence*

$$\Delta C_t^* \triangleq \Delta \ln \overline{V}_t - (\gamma_t^*, \Delta S_t) \geq 0, \quad C_0^* = 0 \quad \mathsf{Q} - a.s., \tag{11}$$

is Q-a.s. nondecreasing and the following decomposition holds for any $\mathsf{Q} \in \Re_N$:

$$f_N\left(S_\bullet\right) = \ln \overline{V}_0 + \sum_{i=1}^{N} \left(\gamma_i^*, \Delta S_i\right) - C_N^* \quad \mathsf{Q} - a.s. \tag{12}$$

Remark 2 *(1) If $Y_t \triangleq \operatorname*{ess\,sup}_{\mathsf{Q} \in \Re_N \cap \mathfrak{M}_N} \mathsf{E}^{\mathsf{Q}} \left[f | \mathcal{F}_t^S \right]$, where f is a $\mathcal{F}_N^S$-measurable bounded random variable, then in [17] (see theorem on page 674), it is proved that $\left\{ Y_t, \mathcal{F}_t^S \right\}_{t \in N_0}$ is a supermartingale with respect to any $\mathsf{Q} \in \Re_N \cap \mathfrak{M}_N$.*

(2) According to [10] (see theorem 7.5 on page 330) the following assertions are equivalent:

(i) $\left\{ Y_t, \mathcal{F}_t^S \right\}_{t \in N_0}$ is a supermartingale with respect to any $\mathsf{Q} \in \Re_N \cap \mathfrak{M}_N$;

(ii) there are nondecreasing sequence $\{C_t^*\}_{t\in N_0}$ *and d-dimensional predictable sequence* $\{\gamma_t^*\}_{t\in N_1}$ *such that* Y_t *admits representation* $Y_t = Y_0 + \sum\limits_{i=1}^{t}(\gamma_i^*, \Delta S_i) - C_t^*$ P*-a.s. This representation is called optional decomposition or uniform Doob decomposition [5, 8, 9, 10, 12].*

In contract to above mentioned works, Theorem 4:

(i) does not require sequence $\left\{\ln \overline{V}_t, \mathcal{F}_t^S\right\}_{t\in N_0}$ *to be a supermartingale with respect to any* $\mathsf{Q}\in\Re_N\cap\mathfrak{M}_N$, $\Re_N\cap\mathfrak{M}_N\neq\varnothing$*;*

(ii) presents constructive method which allows construction of d-dimensional predictable minimax strategy γ_1^{*N} *and nondecreasing sequence* $\{C_t^*\}_{t\in N_0}$*, i.e. components of optional decomposition (12);*

We do not use results of [5, 8, 9, 10, 12] to prove Theorem 4.

(3) Theorem 4 implies the following inequality for any measure $\mathsf{Q}\in\Re_N$*:*

$$f_N(S_\bullet) \le \ln \overline{V}_0 + \sum_{i=1}^{N}(\gamma_i^*, \Delta S_i) \quad \mathsf{Q} - a.s.$$

Thus if the sequence $\left\{S_t, \mathcal{F}_t^S\right\}_{t\in N_0}$ *is a local martingale with respect to a measure* Q*, then* $\left\{\ln \overline{V}_t, \mathcal{F}_t^S\right\}_{t\in N_0}$ *and* $\left\{\mathsf{E}^{\mathsf{Q}}\left[f_N(S_\bullet)\,|\mathcal{F}_t^S\right]\right\}_{t\in N_0}$ *are supermartingales with respect to any* $\mathsf{Q}\in\Re_N$.

(4) Condition $\Re_N\cap\mathfrak{M}_N\neq\varnothing$ *means that* $(1,S)$*-market in consideration is incomplete.*

1.6. Theorem below provides (formal) existence conditions for the worst-case probability measure.

Theorem 5 *Let* ξ *be any* $\mathcal{F}_N$*–measurable* P*-a.s. bounded random variable. Then the following is true:*

(1) there exist

probability measure λ *on* $(\Omega,\mathcal{F})$ *such, that* $\lambda\gg\mathsf{Q}$ *for any* $\mathsf{Q}\in\Re_N$*, and*

a set of non-negative $\mathcal{F}$*-measurable random variables* $\{X_k\}_{k\ge1}$ *with: (i)* $\mathsf{E}^{\lambda}X_k = 1$, $k\ge1$*; (ii)* $\sup\limits_{\mathsf{Q}\in\Re}\mathsf{E}^{\mathsf{Q}}\xi = \lim\limits_{k\to\infty}\mathsf{E}^{\lambda}X_k\xi$*;*

(2) if $\{X_k\}_{k\ge1}$ *is a weakly relatively compact sequence in* $L^1(\Omega,\mathcal{F},\lambda)$*, then there exists probability measure* Q^* *on* $(\Omega,\mathcal{F})$*:*

$$\sup_{\mathsf{Q}\in\Re_N}\mathsf{E}^{\mathsf{Q}}\xi = \mathsf{E}^{\mathsf{Q}^*}\xi. \tag{13}$$

Remark 3 *(1) Weakly relatively compactness condition for* $\{X_k\}_{k\ge1}$ *is difficult to verify. Thus the theorem is non-usable. Still it allows us to consider properties of problem (2) solution.*

(2) In contrast to [10], [17], theorem 5 provides sufficient conditions for Lebesgue integral of bounded measurable random variable to attain it's supremum over the set of equivalent probability measures. Yet it is well known [2], [16], that, as a rule, supremum is attained on finitely additive measure. So, expectation is not defined and it is impossible to construct solution for option's pricing problem. That is why theorem 5 is critical for our construction. Note, that [2], [16] present another non-usable conditions for countable additivity of "extremal" measure.

(3) According to Dunford-Pettis theorem [10] requirement of theorem 5 for $\{X_k\}_{k\geq 1}$ *to be weakly relatively compact might be rewritten as requirement for boundedness and uniform integrability in* $L^1(\Omega, \mathcal{F}, \lambda)$. *Moreover, if* $\{X_k\}_{k\geq 1}$ *is weakly closed and convex, then according to James theorem [10] weakly relatively compactness condition for* $\{X_k\}_{k\geq 1}$ *is necessary and sufficient for a Lebesgue integral to attain supremum.*

(4) Obviously, if $\xi(\omega)$ *takes values in final set or in countable (or final) union of compact sets, then: (i) there is* $\omega^* \in \Omega$*:* $\xi(\omega^*) = \sup\limits_{\omega\in\Omega}$*; (ii)* $\sup\limits_{\mathsf{Q}\in\mathfrak{R}} \mathsf{E}^{\mathsf{Q}}\xi$ *is attained on* Q^**:* $\mathsf{Q}^*(\{\omega^*\}) = 1$, $\mathsf{Q}^*(\Omega \setminus \omega^*) = 0$.

1.7. Here we implement Theorems 3 and 5 to gain new recurrent relation for the sequence of upper guaranteed values.

Theorem 6 *Suppose, the assumptions of Theorems 3 and 5 are satisfied. Then* $\left(\overline{V}_t, \mathcal{F}_t^S\right)_{t\in N_1}$ *satisfies the recurrent relation* Q^**-a.s.*

$$\begin{cases} \overline{V}_{t-1} = \mathsf{E}^{\mathsf{Q}^*}\left[\overline{V}_t e^{-(\gamma_t^*, \Delta S_t)} | \mathcal{F}_{t-1}^S\right], \\ \overline{V}_t|_{t=N} = \exp\{f_N(S_\bullet)\}. \end{cases} \tag{14}$$

1.8. From Theorems 4 and 5 an important assertion follows.

Corollary 2 *If* $\left(\overline{V}_t, \mathcal{F}_t^S\right)_{t\in N_1}$ *satisfies the recurrent relation (14), then for any* $t \in N_0$*, decomposition (11) holds with respect to the measure* Q^**, i.e.,*

$$\Delta \ln \overline{V}_t = (\gamma_t^*, \Delta S_t) - \Delta C_t^* \quad \mathsf{Q}^* - a.s. \tag{15}$$

1.9. In this subsection, we give a criterion for the probability measure Q^* to be the worst-case measure.

Definition 8 *Let* $\left\{\overline{\mu}_t, \mathcal{F}_t^S\right\}_{t\in N_0}$ *be defined by formula*

$$\overline{\mu}_t \triangleq \overline{V}_t \exp\left\{-\sum_{i=1}^{t} (\gamma_i^*, \Delta S_i)\right\}, \tag{16}$$

where $\overline{V}_t$ *satisfies recurrent relation (5) and* $\{\gamma_t^*\}_{t\in N_1} \in D_1^N$ *is the minimax strategy defined by (9). The sequence* $\{\overline{\mu}_t, \mathcal{F}_t^S\}_{t\in N_0}$ *is said to be an upper* S*-estimating one.*

Remark 4 *It follows from Theorem 3 (equality (10)) that the upper* S*-estimating sequence is a supermartingale with respect to any measure* $\mathsf{Q} \in \Re_N$.

Theorem 7 *Suppose, the assumptions of Theorem 6 are satisfied. Then the following conditions are equivalent:*

(1) Q^* *is the worst-case probability distribution;*

(2) equality (14) holds for any $t \in N_1$*;*

(3) the upper S*-estimating sequence* $\{\overline{\mu}_t, \mathcal{F}_t^S\}_{t\in N_0}$ *is a martingale with respect to the measure* Q^*.

1.10. The main result of this section follows from Theorems 1–7.

Theorem 8 *Suppose that the assumptions of Theorem 6 are satisfied. Then there exists a solution of minimax problem (2).*

§2 Minimax hedging of a European option in an incomplete market

In this section, we use the results of Section 1 to link problem (2) and the problem of European options' pricing in incomplete markets without transaction costs. We also give existence conditions for the minimal perfect superhedging portfolio (Theorem 10). We use Theorem 6 to formulate the following assertions: (1) the worst-case measure Q^* is a martingale one (Theorem 11); (2) for any bounded contingent claim, there exists an S-representation [17] with respect to Q^* (Theorem 12). Further, we state that the worst-case measure Q^* is discrete and does not belong to $\Re_N$ (Theorem 13). A $(1, S)$-market with respect to Q^* is called the *worst-case complete market*. The corresponding portfolio is called the *minimax hedging one*. Finally, we provide and prove a method for finding the price of an European option in an incomplete market without transaction costs (Theorem 15).

2.1. In this subsection, we recall some concepts of option pricing theory (see [17], [10]); the economic interpretation can be found in [10].

2.1.1 Let $\{S_t, \mathcal{F}_t\}_{t\in N_0}$ be the d-dimensional adapted sequence defined in Subsection 1.1.1. Suppose that this sequence describes the evolution of d

risky assets' prices [17]. We also suppose that there is a riskless asset [17] with zero return and the initial price 1. This collection of assets is called $(1, S)$-*market* [17]. An $\mathcal{F}_N^S$-measurable random variable $f_N(S_\bullet)$ is called a *European contingent claim with maturity* $N \in \mathbb{N}^+$ [17]. Let $\{\beta_t\}_{t\in N_0}$ be an $\mathcal{F}^S$-predictable one-dimensional sequence. Its elements can be interpreted [17] as the quantity of a riskless asset. Let $\{\gamma_t\}_{t\in N_1}$ be an $\mathcal{F}^S$-predictable d-dimensional sequence introduced in Subsection 1.1.2. Note that such a sequence is called a strategy. The ith $\left(i = \overline{1, d}\right)$ component of the vector γ_t represents [17] the quantity of the ith risky asset at a time $t \in N_1$. The sequence of pairs $\pi \triangleq (\beta_t, \gamma_t)_{t\in N_0}$ is called a *portfolio*. The *capital of the portfolio* π at a time $t \in N_0$ [17] in the $\left(1, S^{(1)}, \ldots, S^{(d)}\right)$-market is an $\mathcal{F}_t^S$-measurable random variable X_t^π such that

$$X_t^\pi = \beta_t + (S_t, \gamma_t). \tag{17}$$

The portfolio π is a *self-financing* one [17] if for any $t \in N_1$, P-a.s.

$$\Delta\beta_t + (S_{t-1}, \Delta\gamma_t) = 0. \tag{18}$$

The set of all self-financing portfolios is denoted by SF.

An adapted nondecreasing sequence $C \triangleq \left\{C_t, \mathcal{F}_t^S\right\}_{t\in N_0}$ such that $C_t|_{t=0} = 0$ is called *consumption* [17]. The pair (π, C) is a portfolio with consumption [17]. The capital of a portfolio with consumption (π, C) at a time $t \in N_0$ is denoted by $\widehat{X}_t^{(\pi)}$ and defined by the formula

$$\widehat{X}_t^\pi \triangleq X_t^\pi - C_t. \tag{19}$$

It follows from (17)–(19) that at any time $t \in N_0$, the capital $\widehat{X}_t^\pi$ of a self-financing portfolio with consumption (π, C) admits the representation P-a.s.

$$\widehat{X}_t^\pi = \widehat{X}_0^\pi + \sum_{i=1}^{t} (\gamma_i, \Delta S_i) - C_t. \tag{20}$$

2.1.2. A $(1, S)$-market is said to be arbitrage-free [17] if the following condition holds for the capital of any portfolio $\pi \in SF$: if $\mathsf{P}\left(X_N^\pi \geq 0 | X_0^\pi = 0\right) = 1$, then $\mathsf{P}\left(X_N^\pi = 0 | X_0^\pi = 0\right) = 1$. It is well known [17] that if there is at least one martingale probability measure in a $(1, S)$-market, then the market is arbitrage-free.

2.1.3. Recall [17], that an arbitrage-free $(1, S)$-market is complete with respect to measure $\tilde{\mathsf{Q}} \in \Re_N \cap \mathfrak{M}_N$ if for any bounded $f_N(S_\bullet)$ there is a portfolio $\pi \in SF$ with the capital X_N^π such that $f_N(S_\bullet) = X_N^\pi \quad \tilde{\mathsf{Q}} - \text{a.s.}$

Definition 9 *[17]. A one-dimensional martingale* $\left(\Theta_t, \mathcal{F}_t^S\right)_{t \in N_0}$ *is said to admit an* S*-representation with respect to the* d*-dimensional martingale* $\left(S_t, \mathcal{F}_t^S\right)_{t \in N_0}$ *and the measure* $\tilde{\mathsf{Q}} \in \Re_N \cap \mathfrak{M}_N$ *if there is an* $\mathcal{F}^S$*-predictable* d*-dimensional sequence* $\{\gamma_t\}_{t \in N_1}$ *such that for any* $t \in N_0$,

$$\Theta_t = \Theta_0 + \sum_{i=1}^{t} (\gamma_i, \Delta S_i) \quad \tilde{\mathsf{Q}} - a.s. \tag{21}$$

The following assertion is well known (see [10], [17]).

Theorem 9 *Suppose that* $\tilde{\mathsf{Q}} \in \mathfrak{M}_N \cap \Re_N \, (\neq \varnothing)$. *Then the following conditions are equivalent:*

(1) $(1, S)$*-market is complete;*

(2) $\mathfrak{M}_N \cap \Re_N = \left\{\tilde{\mathsf{Q}}\right\}$*;*

(3) any local martingale $\left(\Theta_t, \mathcal{F}_t^S\right)_{t \in N_0}$ *admits an* S*-representation with respect to the measure* $\tilde{\mathsf{Q}} \in \mathfrak{M}_N \cap \Re_N$.

That is why they say [10], [17], that an arbitrage-free $(1, S)$-market is incomplete if $|\mathfrak{M}_N \cap \Re_N| > 1$.

2.1.4. Generally speaking, the above-defined (see Subsection 2.1.1) $(1, S)$-market is incomplete. Thus, there is a problem of selecting a measure with respect to which one should calculate the price of an European option. As was previously mentioned, we use the minimax approach to solve this problem. This approach allows us to describe a $(1, S)$-market by means of the worst-case probability distribution. There we will need results of Section 1.

2.1.5. A $(1, S)$-market is said to be non-redundant [10], if for any $t \in N_0$ and $\mathsf{Q} \in \Re_N$ $(\gamma_t, \triangle S_t) = 0$ Q-a.s. implies Q-a.s. triviality for γ_t. Note, non-redundant condition is not essential: one can fairly exclude "excessive" assets.

2.2. Now we give the definition of minimax superhedging portfolio with consumption.

Definition 10 *[17]. A self-financing portfolio with consumption* (π, C) *in a* $(1, S)$*-market in the problem of European option pricing with contingent claim* $f_N(S_\bullet)$ *is said to be superhedging with consumption if* $f_N(S_\bullet) \leq \widehat{X}_N^\pi \quad \mathsf{P} - a.s.$

Definition 11 *[17]. A superhedging portfolio with consumption* (π, C) *is a perfect one if*

$$f_N(S_\bullet) = \widehat{X}_N^\pi \quad \mathsf{P} - a.s. \tag{22}$$

Conditions for the existence of a perfect superhedging portfolio with respect to a measure $\mathsf{Q} \in \mathfrak{M}_N \cap \Re_N$ can be found in [17] (see Theorem 2, p. 652) and in [10] (see Theorem 7.13, p. 335).

Definition 12 *A perfect superhedging portfolio with consumption* (π^*, C^*) *is the minimal one if for any other perfect superhedging portfolio with consumption* (π, C) *and for any* $t \in N_0$*, the inequality holds* P*-a.s.:*

$$\widehat{X}_t^{\pi^*} \leq \widehat{X}_t^{\pi}. \tag{23}$$

In this subsection, we give conditions for the existence of the minimal perfect superhedging portfolio with consumption. These conditions are based on Theorem 4. Moreover, the assertion given below links problem (2) to the problem of minimax perfect superhedging portfolio construction for a European option in an incomplete $(1, S)$-market.

Theorem 10 *Consider a* $(1, S)$*-market. Suppose that the assumptions of Theorem 4 are satisfied. Then the minimal perfect superhedging portfolio with consumption* (π^*, C^*) *exists with respect to any* $\mathsf{Q} \in \Re_N$*, namely*

(1) self-financing portfolio $\pi^* = \{\beta_t^*, \gamma_t^*\}_{t \in N_1}$*, where* $\{\gamma_t^*\}_{t \in N_1} \in D_1^N$ *is an admissible predictable sequence satisfying (9), and* $\{\beta_t^*\}_{t \in N_0}$ *is defined by*

$$\begin{cases} \Delta\beta_t^* + (S_{t-1}, \Delta\gamma_t^*) = 0, \\ \beta_t^*|_{t=0} = \beta_0^*, \end{cases} \tag{24}$$

one can choose $\beta_0^* = \ln \overline{V}_0$ *(which can be obtained using (5)) and* $\gamma_0^* = 0$*; for any* $t \in N_0$*, the capital of the portfolio* π^* *can be represented as*

$$X_t^{\pi^*} = \beta_t^* + (\gamma_t^*, S_t) \quad \mathsf{Q} - a.s. \tag{25}$$

(2) for any $t \in N_0$ *and* $\mathsf{Q} \in \Re_N$*, the capital* $\hat{X}_t^{\pi^*}$ *of the superhedging portfolio with consumption* (π^*, C^*) *admits the representation*

$$\hat{X}_t^{\pi^*} = \ln \overline{V}_t \;\; \mathsf{Q} - a.s., \tag{26}$$

where $\overline{V}_t$ *satisfies recurrent relation (5) and the consumption* C_t^* *at any moment* $t \in N_0$ *admits the representation* Q*-a.s.*

$$\begin{cases} \Delta C_t^* = (\gamma_t^*, \Delta S_t) - \Delta\hat{X}_t^{\pi^*} \geq 0, \\ C_t^*|_{t=0} = 0, \end{cases} \tag{27}$$

moreover, the following equalities hold

(a) $\hat{X}_t^{\pi^*} = X_t^{\pi^*} - C_t^*$ Q*-a.s.,*

(b)

$$\hat{X}_t^{\pi^*} = \ln \overline{V}_0 + \sum_{i=1}^{t} (\gamma_i^*, \Delta S_i) - C_t^* \quad \mathsf{Q} - a.s. \tag{28}$$

(3) (π^*, C^*) *is a perfect superhedging portfolio with consumption, i.e.,*

$$\hat{X}_N^{\pi^*} = f_N(S_\bullet) \quad \mathsf{Q} - a.s.; \tag{29}$$

(4) (π^*, C^*) *is the minimal perfect superhedging portfolio with consumption.*

Remark 5 *It is difficult to apply Theorem 10 to calculate the price of an European option in an incomplete market without transaction costs. Indeed, it is necessary to solve recurrent relation (9). This implies the problem of calculating the consumption* $\{C_t^*\}_{t \in N_0}$.

2.3. Now we are going to consider some useful properties of solution for problem (2). Let us start with martingale property for the worst-case measure.

Theorem 11 *Suppose the solution of problem (2) exists. Then* Q^* *is a martingale measure.*

2.4. In this subsection, we give conditions for the existence of the S-representation for any bounded contingent claim with respect to the measure Q^*. These conditions are based on Theorem 4.

Theorem 12 *Fix arbitrary* $\mathcal{F}_N$*-measurable bounded contingent claim* f_N. *Suppose the solution of problem (2) for such* f_N *exists. Then* f_N *admits representation with respect to the measure* Q^*

$$f_N(S_\bullet) = \mathsf{E}^{\mathsf{Q}^*}\left[f_N(S_\bullet) | \mathcal{F}_0^S\right] + \sum_{i=1}^{N} (\gamma_i^*, \Delta S_i) \quad \mathsf{Q}^* - a.s., \tag{30}$$

where $\left\{\gamma_t^*, \mathcal{F}_{t-1}^S\right\}_{t \in N_1}$ *is the d-dimensional predictable sequence satisfing (9).*

Remark 6 *(1) It is possible that* $\mathsf{Q}^* \notin \Re_N$. *Thus, the well-known Lemma 10 of [17] (see p. 611) and Lemma 5.3.9 of [10] don't imply Theorem 11. In Section 4, we use Theorems 4 and 6 to prove that the S-representation exists.*

(2) It follows from Corollary 2 and Theorem 12 that for any $t \in N_1$,

$$\begin{cases} \Delta \ln \overline{V}_t = (\gamma_t^*, \Delta S_t) \\ \ln \overline{V}_t|_{t=0} = \ln \overline{V}_0, \quad \ln \overline{V}_t|_{t=N} = f_N(S_\bullet) \quad \mathsf{Q}^* - a.s. \end{cases} \tag{31}$$

The measure Q^* *is a martingale one. Hence formula (31) yields*

$$\ln \overline{V}_0 = \mathsf{E}^{\mathsf{Q}^*}\left[f_N\left(S_\bullet\right)|\mathcal{F}_0^S\right] \quad \mathsf{Q}^* - a.s. \tag{32}$$

(3) Formula (31) implies, that in non-redundant $(1,S)$*-market* $\{\gamma_t^*\}_{t\in N_1}$ *is unique (i.e. if there is* $\{\tilde{\gamma}_t\}_{t\in N_1}$ *satisfying (31), then* $\tilde{\gamma}_t = \gamma_t^*$ Q^**-a.s. for any* $t \in N_1$*).*

(4) For any $t \in N_0$*, the consumption* C_t^* *is trivial with respect to the measure* Q^**. Therefore,* $\hat{X}_t^{\pi^*} = X_t^{\pi^*}$ Q^**-a.s. Hence we conclude that the capital of the portfolio* $\pi^* \in SF$ *admits the representation* Q^**-a.s.*

$$X_t^{\pi^*} = \ln \overline{V}_0 + \sum_{i=1}^{t}\left(\gamma_i^*, \Delta S_i\right) \quad \mathsf{Q}^* - a.s \tag{33}$$

at any moment $t \in N_1$*. Moreover,*

$$X_t^{\pi^*} = \mathsf{E}^{\mathsf{Q}^*}\left[f_N\left(S_\bullet\right)|\mathcal{F}_t^S\right] = \ln \overline{V}_t \quad \mathsf{Q}^* - a.s. \tag{34}$$

2.5.

Theorem 13 *Suppose* $\{1,S\}$*-market is non-redundant and the solution of problem (2) exists. Then there is worst-case probability measure such, that regular conditional probabilities* $\mathsf{Q}^*\left(\cdot|\mathcal{F}_{n-1}^S\right)$*,* $t \in N_1$*, are discreet and their supports consist of* $d+1$ *affine-independent predictable variables.*

Remark 7 *From Theorem 13 it follows, that in the case of non-redundant market there exists such worst-case discreet measure* Q^**, that there is no martingale probability measure* Q*:* $\mathsf{Q} \neq \mathsf{Q}^*$ *and* $\mathsf{Q} \sim \mathsf{Q}^*$ *(another contradicts to affine-independence of support's elements). There are two important consequences: (1)* $\mathsf{Q}^* \notin \Re_N \cap \mathfrak{M}_N$*, thus* $\mathsf{Q}^* \notin \Re_N$*; (2)* Q^* *specifies complete market.*

2.6. It follows from Theorems 11–13 and Remark 10 that the considered $(1,S)$-market can be identified with a complete market with respect to Q^*. This remark leads to the following definition.

Definition 13 *Fix* $\mathcal{F}_N^S$*-measurable bounded contingent claim* f_N*. A non-redundant* $(1,S)$*-market will be called the worst-case complete for* f_N *if there exist*

a worst-case martingale probability measure Q^**, such that conditions* $\mathsf{Q} \in \mathfrak{M}_N$ *and* $\mathsf{Q} \sim \mathsf{Q}^*$ *imply* $\mathsf{Q}(A) = \mathsf{Q}^*(A)$ *for any* $A \in \mathcal{F}_N^S$*, and*

a portfolio $\pi^* \in SF$

such that the following equality holds at the moment N*:*

$$X_N^{\pi^*} = f_N \quad \mathsf{Q}^* - a.s.$$

Such a portfolio π^* *is called a minimax hedging portfolio.*

Definition 13 and Theorems 10–13 imply the following assertion.

Theorem 14 *Suppose the solution of problem (2) exists. Then the worst-case complete market exists. Moreover, the capital of the minimax hedging portfolio is equal to the capital of the minimal perfect superhedging portfolio with* Q^**-a.s. trivial consumption*

2.7. It follows from Theorem 10 that there exists a minimal perfect superhedging portfolio with respect to any measure $\mathsf{Q} \in \Re_N$. However, this result does not provide any method for constructing such a portfolio and its capital. In this subsection, we give a general representation of the minimax hedging portfolio and of its capital by use of Theorems 11–14.

Theorem 15 *Let* $\pi^* \in SF$ *be the minimax hedging portfolio. Suppose the solution of problem (2) exists. Then:*

(1) the minimax hedging portfolio π^* *admits the following representation:*

(i) for each $t \in N_1$*, there exists an* $\mathcal{F}^S_{t-1}$*-measurable d-dimensional* γ^*_t *such that*

$$\begin{cases} \overline{V}_{t-1} = \operatorname*{ess\,inf}_{\gamma \in D_t} \mathsf{E}^{\mathsf{Q}^*}\left[\overline{V}_t e^{-(\gamma,\Delta S_t)} | \mathcal{F}^S_{t-1}\right] = \mathsf{E}^{\mathsf{Q}^*}\left[\overline{V}_t e^{-(\gamma^*_t,\Delta S_t)} | \mathcal{F}^S_{t-1}\right] \quad \mathsf{Q}^* - a.s. \\ \overline{V}_t|_{t=N} = \exp\left\{f_N\left(S_\bullet\right)\right\}; \end{cases} \tag{35}$$

(ii) for each $t \in N_1$*, there exists an* $\mathcal{F}^S_{t-1}$*-measurable* β^*_t *such that*

$$\begin{cases} \beta^*_t = \beta^*_{t-1} - (\gamma^*_t, S_{t-1}) \\ \beta^*_t|_{t=0} = \beta^*_0, \end{cases} \tag{36}$$

where $\beta^*_0 = \ln \overline{V}_0$ *and* $\gamma^*_0 = 0$*;*

(2) for each $t \in N_0$*, the capital* $X^{\pi^*}_t$ *of* $\pi^* \in SF$ *admits the following representations* Q^**-a.s.:*

(a) $X^{\pi^*}_t = \ln \overline{V}_t$*,*

(b) $X^{\pi^*}_t = X^{\pi^*}_0 + \sum_{i=1}^{t} (\gamma^*_i, \Delta S_i)$*,* $X^{\pi^*}_N = f_N(S_\bullet)$ Q^**-a.s.,* $X^{\pi^*}_0 = \ln \overline{V}_0 = \mathsf{E}^{\mathsf{Q}^*}\left[f_N(S_\bullet) | \mathcal{F}^S_0\right]$.

Remark 8 *The capital of the minimax hedging portfolio is* $X^{\pi^*}_0 = \mathsf{E}^{\mathsf{Q}^*}\left[f_N(S_\bullet) | \mathcal{F}^S_0\right]$*. The value* $X^{\pi^*}_0$ *is the upper bound of the spread for an European option in an incomplete market.*

§3. Proofs of Assertions 1–9

3.1. Here we prove Theorem 1 and Corollary 1. To prove these assertions, we make some preparatory notes.

3.1.1. Since $\mathsf{Q} \in \Re_N$ is a measure, it follows from the Radon–Nikodym theorem that there exists a unique $\mathcal{F}_N$-measurable positive random variable z_N such that $z_N(\omega) = \frac{d\mathsf{Q}}{d\mathsf{P}}(\omega)$. The variable z_N is called the density of the measure Q with respect to measure P. Suppose that $\mathsf{Q}_t \triangleq \mathsf{Q}|_{\mathcal{F}_t}$ and $\mathsf{P}_t \triangleq \mathsf{P}|_{\mathcal{F}_t}$. Obviously, for any $t \in N_0$, the probability measures Q_t and P_t are equivalent. Hence there exists a unique $\mathcal{F}_t$-measurable positive random variable $z_t(\omega) \triangleq \frac{d\mathsf{Q}_t}{d\mathsf{P}_t}(\omega)$ such that:

(i) for any $t \in N_0$, $0 < z_t < \infty$ P-a.s.;

(ii) if $\mathsf{Q}_0 = \mathsf{P}_0$, then $z_t|_{t=0} = 1$;

(iii) for each $t \in N_1$, $\mathsf{E}^{\mathsf{P}}(z_t|\mathcal{F}_{t-1}) = z_{t-1}$ P-a.s. The variable $z_t(\omega)$ is a local density.

For any $t \in N_0$, we let $\overline{Z}_t^N$ denote the set of sequences $\left\{\overline{z}_s^{t,N}, \mathcal{F}_s^S\right\}_{s\in N_0}$ such that

$$\overline{z}_s^{t,N} \triangleq \begin{cases} 1, & 0 \le s \le t, \\ \frac{z_s}{z_t}, & t < s \le N. \end{cases} \qquad (37)$$

Let us denote $\overline{z}_t^N \triangleq \left\{\overline{z}_s^{t,N}\right\}_{s=N}$.

It is clear that the sequence $\left\{\overline{z}_s^{t,N}\right\}_{s\in N_0}$ is a martingale with respect to the measure P and the filtration $\left\{\mathcal{F}_s^S\right\}_{s\in N_0}$.

The family of sets $\left\{\overline{Z}_t^N\right\}_{t\in N_0}$ has the following properties (see [17]):

(1) $\overline{Z}_t^N \subseteq \overline{Z}_{t-1}^N \subseteq \cdots \subseteq \overline{Z}_0^N$;

(2) for each $t \in N_0$, a set $\overline{Z}_t^N$ is convex.

The reduction of the set $\overline{Z}_0^N$ to $\{t_1, \ldots, t_2\}$, where $t_1 < t_2$ and $t_1, t_2 \in N_0$, is denoted by $\overline{Z}_{t_1}^{t_2}$, and its elements are denoted by $\overline{z}_{t_1}^{t_2}$.

3.1.2. Consider the estimate of an admissible t-bistrategy $\left(\mathsf{Q}, \gamma_{t+1}^N\right)$. It follows from the law of iterated expectations that $I_t^{\mathsf{Q},\gamma_{t+1}^N}(S_0^t)$ satisfies the recurrent relation Q-a.s.

$$\begin{cases} I_t^{\mathsf{Q},\gamma_{t+1}^N}(S_0^t) = \mathsf{E}^{\mathsf{Q}}\left[I_{t+1}^{\mathsf{Q},\gamma_{t+2}^N}\left(S_0^{t+1}\right)e^{-(\gamma_{t+1},\Delta S_{t+1})}|\mathcal{F}_t^S\right], \\ I_t^{\mathsf{Q},\gamma_{t+1}^N}(S_0^t)|_{t=N} = \exp\left\{f_N(S_\bullet)\right\}. \end{cases} \qquad (38)$$

As we know, Q, $\mathsf{P} \in \Re_N$. Therefore, it follows from the definition of the estimate of an admissible t-bistrategy $\left(\mathsf{Q}, \gamma_{t+1}^N\right)$ (denoted by $I_t^{\mathsf{Q},\gamma_{t+1}^N}(S_0^t)$) and

Girsanov's theorem [18] that Q (P)-a.s.

$$I_t^{\mathsf{Q},\gamma_{t+1}^N}\left(S_0^t\right) = \mathsf{E}^{\mathsf{P}}\left[\overline{z}_t^N \exp\left\{f_N\left(S_\bullet\right) - \sum_{i=t+1}^{N}\left(\gamma_i, \Delta S_i\right)\right\}|\mathcal{F}_t^S\right]. \tag{39}$$

Let us also denote the right-hand side of equality (39) by $I_t^{\mathsf{P},\overline{z}_t^N,\gamma_{t+1}^N}\left(S_0^t\right)$, where $\left(\overline{z}_t^N, \gamma_{t+1}^N\right) \in \overline{Z}_t^N \times D_{t+1}^N$. It follows from (38) and (39) that for any $t \in N_0$ and $\left(\overline{z}_t^N, \gamma_{t+1}^N\right) \in \overline{Z}_t^N \times D_{t+1}^N$, the random variable $I_t^{\mathsf{P},\overline{z}_t^N,\gamma_{t+1}^N}\left(S_0^t\right)$ satisfies the recurrent relation P-a.s.

$$I_t^{\mathsf{P},\overline{z}_t^N,\gamma_{t+1}^N}\left(S_0^t\right) = \mathsf{E}^{\mathsf{P}}\left[\overline{z}_{t+1}^{t,N} I_{t+1}^{\mathsf{P},\overline{z}_{t+1}^N,\gamma_{t+2}^N}\left(S_0^{t+1}\right) e^{-\left(\gamma_{t+1},\Delta S_{t+1}\right)}|\mathcal{F}_t^S\right]. \tag{40}$$

For any $t \in N_0$, we have $I_t^{\mathsf{Q},\gamma_{t+1}^N}\left(S_0^t\right) = I_t^{\mathsf{P},\overline{z}_t^N,\gamma_{t+1}^N}\left(S_0^t\right)$ P-a.s. Hence the random variable $\overline{V}_t$ admits the representation

$$\overline{V}_t = \operatorname*{ess\,inf}_{\gamma_{t+1}^N \in D_{t+1}^N} \operatorname*{ess\,sup}_{\overline{z}_t^N \in \overline{Z}_t^N} I_t^{\mathsf{P},\overline{z}_t^N,\gamma_{t+1}^N}\left(S_0^t\right) \quad \mathsf{P}-\text{a.s.} \tag{41}$$

3.1.3. Proof of Theorem 1. First we prove that for any $t \in N_0$, the following inequality holds P-a.s.:

$$\overline{V}_t \geq \operatorname*{ess\,inf}_{\gamma \in D_{t+1}} \operatorname*{ess\,sup}_{\mathsf{Q} \in \Re_N} \mathsf{E}^{\mathsf{Q}}\left[\overline{V}_{t+1} e^{-\left(\gamma,\Delta S_{t+1}\right)}|\mathcal{F}_t^S\right]. \tag{42}$$

By definition,

$$\overline{I}_t^{\mathsf{P},\gamma_{t+1}^N}\left(S_0^t\right) \triangleq \operatorname*{ess\,sup}_{\overline{z}_t^N \in \overline{Z}_t^N} I_t^{\mathsf{P},\overline{z}_t^N,\gamma_{t+1}^N}\left(S_0^t\right). \tag{43}$$

It follows from the definition of essential supremum that $\overline{I}_t^{\mathsf{P},\gamma_{t+1}^N}\left(S_0^t\right)$ is an $\mathcal{F}_t^S$-measurable variable. The $I_t^{\mathsf{P},\overline{z}_t^N,\gamma_{t+1}^N}\left(S_0^t\right)$ satisfies recurrent relation (40). Hence the properties of essential supremum and Girsanov's theorem imply the formulas P-a.s.

$$\begin{aligned}
\overline{I}_t^{\mathsf{P},\gamma_{t+1}^N}\left(S_0^t\right) &= \operatorname*{ess\,sup}_{\overline{z}_t^N \in \overline{Z}_t^N} \mathsf{E}^{\mathsf{P}}\left[\overline{z}_{t+1}^{t,N} I_{t+1}^{\mathsf{P},\overline{z}_{t+1}^N,\gamma_{t+2}^N}\left(S_0^{t+1}\right) e^{-\left(\gamma_{t+1},\Delta S_{t+1}\right)}|\mathcal{F}_t^S\right] \geq \\
&\geq \operatorname*{ess\,sup}_{\overline{z}_{t+1}^N \in \overline{Z}_{t+1}^N} \mathsf{E}^{\mathsf{P}}\left[\overline{z}_{t+1}^{t,N} I_{t+1}^{\mathsf{P},\overline{z}_{t+1}^N,\gamma_{t+2}^N}\left(S_0^{t+1}\right) e^{-\left(\gamma_{t+1},\Delta S_{t+1}\right)}|\mathcal{F}_t^S\right] = \\
&= \mathsf{E}^{\mathsf{P}}\left[\overline{z}_{t+1}^{t,N} \operatorname*{ess\,sup}_{\overline{z}_{t+1}^N \in \overline{Z}_{t+1}^N} I_{t+1}^{\mathsf{P},\overline{z}_{t+1}^N,\gamma_{t+2}^N}\left(S_0^{t+1}\right) e^{-\left(\gamma_{t+1},\Delta S_{t+1}\right)}|\mathcal{F}_t^S\right] = \\
&= \mathsf{E}^{\mathsf{P}}\left[\overline{z}_{t+1}^{t,N} \overline{I}_{t+1}^{\mathsf{P},\gamma_{t+2}^N}\left(S_0^{t+1}\right) e^{-\left(\gamma_{t+1},\Delta S_{t+1}\right)}|\mathcal{F}_t^S\right] =
\end{aligned} \tag{44}$$

$$= \mathsf{E}^{\mathsf{Q}}\left[\overline{I}_{t+1}^{\mathsf{P},\gamma_{t+2}^{N}}\left(S_0^{t+1}\right)e^{-(\gamma_{t+1},\Delta S_{t+1})}|\mathcal{F}_t^S\right].$$

Note that the left-hand side of (44) does not depend on the measure Q. Hence, it follows from (44) that P-a.s.

$$\overline{I}_t^{\mathsf{P},\gamma_{t+1}^{N}}\left(S_0^{t}\right) \geq \operatorname*{ess\,sup}_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}}\left[\overline{I}_{t+1}^{\mathsf{P},\gamma_{t+2}^{N}}\left(S_0^{t+1}\right)e^{-(\gamma_{t+1},\Delta S_{t+1})}|\mathcal{F}_t^S\right].$$

This inequality can be sharpened as follows:

$$\overline{I}_t^{\mathsf{P},\gamma_{t+1}^{N}}\left(S_0^{t}\right) \geq \operatorname*{ess\,sup}_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}}\left[\operatorname*{ess\,inf}_{\gamma_{t+2}^N\in D_{t+2}^N}\overline{I}_{t+1}^{\mathsf{P},\gamma_{t+2}^{N}}\left(S_0^{t+1}\right)e^{-(\gamma_{t+1},\Delta S_{t+1})}|\mathcal{F}_t^S\right] \geq \tag{45}$$

$$\geq \operatorname*{ess\,inf}_{\gamma_{t+1}\in D_{t+1}}\operatorname*{ess\,sup}_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}}\left[\operatorname*{ess\,inf}_{\gamma_{t+2}^N\in D_{t+2}^N}\overline{I}_{t+1}^{\mathsf{P},\gamma_{t+2}^{N}}\left(S_0^{t+1}\right)e^{-(\gamma_{t+1},\Delta S_{t+1})}|\mathcal{F}_t^S\right]. \quad \mathsf{P}\text{-a.s.}$$

Applying the formula $\overline{V}_{t+1} = \operatorname*{ess\,inf}_{\gamma_{t+2}^N\in D_{t+2}^N}\overline{I}_{t+1}^{\mathsf{P},\gamma_{t+2}^{N}}\left(S_0^{t+1}\right)$ to inequality (45), we obtain P-a.s.

$$\overline{I}_t^{\mathsf{P},\gamma_{t+1}^{N}}\left(S_0^{t}\right) \geq \operatorname*{ess\,inf}_{\gamma_{t+1}\in D_{t+1}}\operatorname*{ess\,sup}_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}}\left[\overline{V}_{t+1}e^{-(\gamma_{t+1},\Delta S_{t+1})}|\mathcal{F}_t^S\right]. \tag{46}$$

The right-hand side of (46) does not depend on $\gamma_{t+1}\in D_{t+1}$. So (46) imply (42).

Now let us show that for any $t\in N_0$, the following inequality holds P-a.s.:

$$\overline{V}_t \leq \operatorname*{ess\,inf}_{\gamma\in D_{t+1}}\operatorname*{ess\,sup}_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}}\left[\overline{V}_{t+1}e^{-(\gamma,\Delta S_{t+1})}|\mathcal{F}_t^S\right]. \tag{47}$$

Since $I_{t+1}^{\mathsf{P},\overline{z}_{t+1}^N,\gamma_{t+2}^N}\left(S_0^{t+1}\right) \leq \overline{I}_{t+1}^{\mathsf{P},\gamma_{t+2}^{N}}\left(S_0^{t+1}\right)$ P-a.s., it follows from Girsanov's theorem that for any $\gamma_{t+1}^N\in D_{t+1}^N$ the inequalities hold P-a.s.

$$I_t^{\mathsf{P},\overline{z}_t^N,\gamma_{t+1}^N}\left(S_0^{t}\right) \leq \mathsf{E}^{\mathsf{Q}}\left[\overline{I}_{t+1}^{\mathsf{P},\gamma_{t+2}^{N}}\left(S_0^{t+1}\right)e^{-(\gamma_{t+1},\Delta S_{t+1})}|\mathcal{F}_t^S\right] \leq \tag{48}$$

$$\leq \operatorname*{ess\,sup}_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}}\left[\overline{I}_{t+1}^{\mathsf{P},\gamma_{t+2}^{N}}\left(S_0^{t+1}\right)e^{-(\gamma_{t+1},\Delta S_{t+1})}|\mathcal{F}_t^S\right].$$

The right-hand side of (48) does not depend on Q. This implies that for any $\gamma_{t+1}^N\in D_{t+1}^N$, the following inequality holds P-a.s.:

$$\overline{I}_t^{\mathsf{P},\gamma_{t+1}^{N}}\left(S_0^{t}\right) \leq \operatorname*{ess\,sup}_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}}\left[\overline{I}_{t+1}^{\mathsf{P},\gamma_{t+2}^{N}}\left(S_0^{t+1}\right)e^{-(\gamma_{t+1},\Delta S_{t+1})}|\mathcal{F}_t^S\right]. \tag{49}$$

Note that: (1) for any $\gamma_{t+1}^N \in D_{t+1}^N$, we have $\overline{V}_t \leq \overline{I}_t^{\mathsf{P},\gamma_{t+1}^N}(S_0^t)$ P-a.s.; (2) it follows from the definition of essential infimum that for any $\varepsilon > 0$, there exists $\overline{\gamma}_{t+1}^{\varepsilon,N} \triangleq \{\overline{\gamma}_s^\varepsilon\}_{s\in\{t+1,\dots,N\}} \in D_{t+1}^N$, where $\overline{\gamma}_s^\varepsilon$ is an $\mathcal{F}_{s-1}^S$-measurable d-dimensional vector $\overline{\gamma}_{t+1}^{\varepsilon,N}$ (which depends on ε) such that for any $t \in N_0$,

$$\overline{V}_t \geq \overline{I}_t^{\mathsf{P},\overline{\gamma}_{t+1}^{\varepsilon,N}}\left(S_0^t\right) - \varepsilon \quad \mathsf{P}-\text{a.s.}$$

Thus (49) can be rewritten as

$$\begin{aligned}\overline{V}_t &\leq \overline{I}_t^{\mathsf{P},\overline{\gamma}_{t+1}^{\varepsilon,N}}\left(S_0^t\right) \leq \operatorname*{ess\,sup}_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}}\left[\overline{I}_{t+1}^{\mathsf{P},\overline{\gamma}_{t+2}^{\varepsilon,N}}\left(S_0^{t+1}\right)e^{-(\gamma_{t+1},\Delta S_{t+1})}|\mathcal{F}_t^S\right] \leq \\ &\leq \operatorname*{ess\,sup}_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}}\left[\left(\overline{V}_{t+1}+\varepsilon\right)e^{-(\gamma_{t+1},\Delta S_{t+1})}|\mathcal{F}_t^S\right] \leq \\ &\leq \operatorname*{ess\,sup}_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}}\left[\overline{V}_{t+1}e^{-(\gamma_{t+1},\Delta S_{t+1})}|\mathcal{F}_t^S\right] + \\ &\quad +\varepsilon \operatorname*{ess\,sup}_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}}\left[e^{-(\gamma_{t+1},\Delta S_{t+1})}|\mathcal{F}_t^S\right]. \end{aligned} \tag{50}$$

We consider the second term in the left-hand side of (50). As $\gamma_{t+1} \in D_{t+1}$, we have for any $t \in N_1$,

$$0 \leq \operatorname*{ess\,sup}_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}}\left[e^{-(\gamma_{t+1},\Delta S_{t+1})}|\mathcal{F}_t^S\right] < \infty.$$

The constant $\varepsilon > 0$ is arbitrary. Hence the following inequality holds for any $\gamma_{t+1} \in D_{t+1}$ and $t \in N_1$:

$$\overline{V}_t \leq \operatorname*{ess\,sup}_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}}\left[\overline{V}_{t+1}e^{-(\gamma_{t+1},\Delta S_{t+1})}|\mathcal{F}_t^S\right]. \tag{51}$$

As the left-hand side of (51) does not depend on $\gamma_{t+1} \in D_{t+1}$, we obtain (47).

Inequalities (42), (47) imply recurrent relation (5). Obviously, $\overline{V}_t|_{t=N} = e^{f_N(S_\bullet)}$. This completes the proof of the theorem.

3.1.4. PROOF OF COROLLARY 1. (1) For any $\mathsf{Q} \in \Re_N$, we have

$$\operatorname*{ess\,sup}_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}}\left[\overline{V}_t e^{-(\gamma,\Delta S_t)}|\mathcal{F}_{t-1}^S\right] \geq \mathsf{E}^{\mathsf{Q}}\left[\overline{V}_t e^{-(\gamma,\Delta S_t)}|\mathcal{F}_{t-1}^S\right] \quad \mathsf{P}-\text{a.s.}$$

Therefore for any $t \in N_0$, $\gamma \in D_t$ and $\mathsf{Q} \in \Re_N$, the following inequality holds

$$\operatorname*{ess\,inf}_{\gamma\in D_t} \operatorname*{ess\,sup}_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}}\left[\overline{V}_t e^{-(\gamma,\Delta S_t)}|\mathcal{F}_{t-1}^S\right] \geq \operatorname*{ess\,inf}_{\gamma\in D_t} \mathsf{E}^{\mathsf{Q}}\left[\overline{V}_t e^{-(\gamma,\Delta S_t)}|\mathcal{F}_{t-1}^S\right]. \quad \mathsf{P}-\text{a.s.}$$

The combination of (5) and of the last inequality implies (6).

(2) The second inequality immediately follows from (5) and the definition of essential supremum. This completes the proof of the corollary.

3.2. PROOF OF THEOREM 2. Let us prove that for any $t \in N_1$, we have

$$\overline{V}_t \le e^{c_2} \quad \mathsf{P}-\text{a.s.} \tag{52}$$

We proceed by induction. It obviously follows from the conditions of Theorem 2 that

$$\overline{V}_t|_{t=N} \le e^{c_2}.$$

To prove (52) it is necessary to show that if $\overline{V}_t \le e^{c_2}$, then $\overline{V}_{t-1} \le e^{c_2}$ for any $t \in N_1$. Suppose that $\overline{V}_t \le e^{c_2}$. It follows from Corollary 1 that for any $\gamma \in D_t$ we have P-a.s.

$$\overline{V}_{t-1} = \operatorname*{ess\,inf}_{\gamma \in D_t} \operatorname*{ess\,sup}_{\mathsf{Q} \in \Re_N} \mathsf{E}^{\mathsf{Q}} \left[\overline{V}_t e^{-(\gamma, \Delta S_t)} | \mathcal{F}^S_{t-1} \right] \le$$

$$\le \operatorname*{ess\,sup}_{\mathsf{Q} \in \Re_N} \mathsf{E}^{\mathsf{Q}} \left[\overline{V}_t e^{-(\gamma, \Delta S_t)} | \mathcal{F}^S_{t-1} \right].$$

We can sharpen the last inequality because $0 \in D_t$. We have P-a.s.

$$\overline{V}_{t-1} \le \operatorname*{ess\,sup}_{\mathsf{Q} \in \Re_N} \mathsf{E}^{\mathsf{Q}} \left[\overline{V}_t | \mathcal{F}^S_{t-1} \right] \le e^{c_2}.$$

So, the main step of induction is proved, i.e., it is proved that (52) is true for any $t \in N_1$.

Now let us show that for any $t \in N_1$,

$$\overline{V}_t \ge e^{-c_2} \quad \mathsf{P}-\text{a.s.} \tag{53}$$

We again proceed by induction, and it obviously follows from the conditions of the theorem that

$$\overline{V}_t|_{t=N} \ge e^{-c_2} \quad \mathsf{P}-\text{a.s.}$$

Hence it is necessary to prove that if $\overline{V}_t \ge e^{-c_2}$, then $\overline{V}_{t-1} \ge e^{-c_2}$ for any $t \in N_1$. Suppose that $\overline{V}_t \ge e^{-c_2}$. It follows from Corollary 1 that P-a.s.

$$\overline{V}_{t-1} = \operatorname*{ess\,inf}_{\gamma \in D_t} \operatorname*{ess\,sup}_{\mathsf{Q} \in \Re_N} \mathsf{E}^{\mathsf{Q}} \left[\overline{V}_t e^{-(\gamma, \Delta S_t)} | \mathcal{F}^S_{t-1} \right] \ge$$

$$\ge \operatorname*{ess\,inf}_{\gamma \in D_t} \mathsf{E}^{\mathsf{Q}} \left[\overline{V}_t e^{-(\gamma, \Delta S_t)} | \mathcal{F}^S_{t-1} \right] \ge \operatorname*{ess\,inf}_{\gamma \in D_t} \mathsf{E}^{\mathsf{Q}} \left[e^{-c_2} e^{-(\gamma, \Delta S_t)} | \mathcal{F}^S_{t-1} \right] =$$

$$= e^{-c_2} \operatorname*{ess\,inf}_{\gamma \in D_t} \mathsf{E}^{\mathsf{Q}} \left[e^{-(\gamma, \Delta S_t)} | \mathcal{F}^S_{t-1} \right]. \tag{54}$$

We shall need the following notation. Let

$$G_{\mathsf{Q}} \left(t, S_0^{t-1}, -\gamma \right) \triangleq \ln \mathsf{E}^{\mathsf{Q}} \left[\exp \left\{ - \left(\gamma, \Delta S_t \right) \right\} | \mathcal{F}^S_{t-1} \right]$$

for any $t \in N_1$, $\gamma \in \mathbb{R}^d$, and $\mathsf{Q} \in \Re_N$. $G_{\mathsf{Q}}\left(t, S_0^{t-1}, -\gamma\right)$ is called the *cumulant* of a random variable ΔS_t with respect to a probability measure Q and the σ-algebra $\mathcal{F}_{t-1}$ (see [17], [18]).

Thence (54) can be rewritten as P-a.s.

$$\overline{V}_{t-1} \geq e^{-c_2} \operatorname*{ess\,inf}_{\gamma \in D_t} \mathsf{E}^{\mathsf{Q}}\left[e^{-(\gamma,\Delta S_t)} | \mathcal{F}_{t-1}^S\right] = e^{-c_2} \operatorname*{ess\,inf}_{\gamma \in D_t} e^{G_{\mathsf{Q}}\left(t, S_0^{t-1}, -\gamma\right)}, \qquad (55)$$

where $\mathsf{Q} \in \Re_N$ is arbitrary.

According to the conditions of the theorem, we have $\Re_N \cap \mathfrak{M}_N \neq \varnothing$. So there exists a measure $\widetilde{\mathsf{Q}} \in \Re_N \cap \mathfrak{M}_N$ such that for any $\gamma \in D_t$ the cumulant satisfies the condition $G_{\widetilde{\mathsf{Q}}}\left(t, S_0^{t-1}, -\gamma\right) \geq 0$ and is a convex function of γ. Hence it follows from (55) that

$$\overline{V}_{t-1} \geq e^{-c_2} \quad \mathsf{P}-\text{a.s.}$$

Consequently, the main step of induction is proved. Also, (53) is proved for any $t \in N_1$. Thus the proof of item 2 of the lemma follows from (53) and (54). The proof is complete.

3.3. To prove Theorem 3 we shall need two auxiliary assertions.

3.3.1.

Lemma 1 *Suppose the following conditions are satisfied:*

(1) $\mathsf{Q} \in \Re_N \cap \mathfrak{M}_N$*;*

(2) $\{S_t\}_{0 \leq t \leq N}$ *is a sequence of d-dimensional variables, where for any* $t \in N_1$ *variables* $\Delta S_n^{(i)}$ *are linearly independent,* $i = \overline{1, d}$*;*

(3) γ_t *is a nontrivial bounded d-dimensional vector.*

Then it is Q*-a.s. true that*

$$\mathsf{Q}\left((\gamma_t, \Delta S_t) > 0 | \mathcal{F}_{t-1}^S\right) > 0, \quad \mathsf{Q}\left((\gamma_t, \Delta S_t) < 0 | \mathcal{F}_{t-1}^S\right) > 0. \qquad (56)$$

Proof of Lemma 1. According to conditions of the Lemma the measure Q is a martingale one. Therefore for any $\gamma_t' \in \mathbb{R}^d$ we have inequality $1 \leq \exp\left\{G_{\mathsf{Q}}\left(t, S_0^{t-1}, -\gamma_t'\right)\right\} = \mathsf{E}^{\mathsf{Q}}\left[e^{-(\gamma_t',\Delta S_t)} | \mathcal{F}_{t-1}\right]$ Q-a.s., where $G_{\mathsf{Q}}\left(t, S_0^{t-1}, -\gamma_t'\right)$ is a cumulant with respect to the measure Q. A cumulant G_{Q} is a convex eigenfunction. So, there exists $\gamma_t \in \mathbb{R}^d$ such that $\exp\left\{G_{\mathsf{Q}}\left(t, S_0^{t-1}, -\gamma_t\right)\right\} > 1$ Q-a.s.

Note that Q-a.s.

$$\begin{aligned} e^{-(\gamma_t,\Delta S_t)} = e^{-(\gamma_t,\Delta S_t)}\left[1_{\{(\gamma_t,\Delta S_t)<0\}} + 1_{\{(\gamma_t,\Delta S_t)\geq 0\}}\right] \leq \\ \leq e^{-(\gamma_t,\Delta S_t)} 1_{\{(\gamma_t,\Delta S_t)<0\}} + 1. \end{aligned}$$

Thus, $1 < \mathsf{E}^{\mathsf{Q}}\left[1 + e^{-(\gamma_t,\Delta S_t)} 1_{\{(\gamma_t,\Delta S_t)<0\}} | \mathcal{F}_{t-1}\right]$ Q-a.s., which can be rewritten as follows Q-a.s.

$$0 < \mathsf{E}^{\mathsf{Q}}\left[e^{-(\gamma_t,\Delta S_t)} 1_{\{(\gamma_t,\Delta S_t)<0\}} | \mathcal{F}_{t-1}\right]. \qquad (57)$$

Let us define $\tilde{\mathsf{Q}}(A) \triangleq \mathsf{E}^{\mathsf{Q}} 1_A(\omega) \exp\left\{\sum_{i=1}^{N}\left[-(\gamma_i, \Delta S_i) - G_{\mathsf{Q}}\left(i, S_0^{i-1}, -\gamma_i\right)\right]\right\}$, where $A \in \mathcal{F}_N^S$ is arbitrary. Obviously, $\tilde{\mathsf{Q}} \sim \mathsf{Q}$. Then due to Girsanov's theorem inequality (57) takes the form Q-a.s.

$$\begin{aligned} 0 < \exp\left\{G_{\mathsf{Q}}\left(t, S_0^{t-1}, -\gamma_t\right)\right\} \mathsf{E}^{\tilde{\mathsf{Q}}}\left[1_{\{(\gamma_t, \Delta S_t) < 0\}} | \mathcal{F}_{t-1}\right] = \\ = \exp\left\{G_{\mathsf{Q}}\left(t, S_0^{t-1}, -\gamma_t\right)\right\} \tilde{\mathsf{Q}}\left[(\gamma_t, \Delta S_t) < 0 | \mathcal{F}_{t-1}\right]. \end{aligned}$$

It means that $0 < \tilde{\mathsf{Q}}\left[(\gamma_t, \Delta S_t) < 0 | \mathcal{F}_{t-1}\right]$ Q-a.s. As measures Q and $\tilde{\mathsf{Q}}$ are equivalent, we have $0 < \mathsf{Q}\left[(\gamma_t, \Delta S_t) < 0 | \mathcal{F}_{t-1}\right]$ Q-a.s.

The same reasoning proves, that $0 < \mathsf{Q}\left[(\gamma_t, \Delta S_t) > 0 | \mathcal{F}_{t-1}\right]$ Q-a.s. The proof is complete.

3.3.2.

Remark 9 *(1) Condition (2) of Lemma 1 is not a crucial one. Indeed, suppose that for some moment $t' \in N_1$ variables $\Delta S_{t'}^{(i)}$, $i = \overline{1, d}$, are not independent and conditional probability $\mathsf{Q}\left((\bar{\gamma}_{t'}, \Delta S_{t'}) = 0 | \mathcal{F}_{t'-1}^S\right) = 1$ Q-a.s. As $\bar{\gamma}_{t'}$ is arbitrary, we have $\Delta S_{t'} = 0$ Q-a.s. Obviously, the last equality means that $\mathcal{F}_{t'}^S = \mathcal{F}_{t'-1}^S$ and $\overline{V}_{t'} = \overline{V}_{t'-1}$ Q-a.s. Hence moment t' might be skipped.*

(2) Suppose the assumptions of Lemma 1 are satisfied and $\bar{\gamma}_t \triangleq \dfrac{\gamma_t}{|\gamma_t|}$, where γ_t is a nontrivial bounded d-dimensional predictable vector. Then any $\mathsf{Q} \in \Re_N \cap \mathfrak{M}_N$ has regular version of conditional distribution $\mathsf{Q}\left((\bar{\gamma}_t, \Delta S_t) \le x | \mathcal{F}_{t-1}^S\right)$. This and (56) means that for any $t \in N_1$ there are positive constants c_3 and c_4 such that Q-a.s.

$$\mathsf{Q}\left(-(\gamma_t, \Delta S_t) \ge c_3 |\gamma_t| \middle| \mathcal{F}_{t-1}^S\right) \ge c_4 > 0. \tag{58}$$

3.3.3. Let us denote

$$\Phi(t, \gamma, \omega) \triangleq \operatorname*{ess\,sup}_{\mathsf{Q} \in \Re_N} \mathsf{E}^{\mathsf{Q}}\left[\overline{V}_t e^{-(\gamma, \Delta S_t)} | \mathcal{F}_{t-1}^S\right], \tag{59}$$

where $t \in N_1$, $\gamma \in D_t$ are arbitrary.

Lemma 2 *Suppose the assumptions of Theorem 2 are satisfied. Then for any $t \in N_1$ the following assertions are true:*

(1) there are positive constants c_3 and c_4 such that Q-a.s.

$$\Phi(t, \gamma, \omega) \ge c_3 e^{c_4 |\gamma| - c_2}; \tag{60}$$

(2) there is version of the function $\Phi(t, \gamma, \omega)$ such that for any $\omega \in \Omega$ this version is convex continuous function of γ.

PROOF OF LEMMA 2. (1) From (59), Corollary 1 and Theorem 2 it follows that for any $\gamma \in D_{t+1}$ and $\mathsf{Q} \in \Re_N \cap \mathfrak{M}_N$ the following inequalities hold Q-a.s.

$$\begin{aligned}\Phi(t,\gamma,\omega) \geq \mathsf{E}^{\mathsf{Q}}\left[\overline{V}_{t+1}e^{-(\gamma,\Delta S_{t+1})}|\mathcal{F}_t^S\right] \geq e^{-c_2}\mathsf{E}^{\mathsf{Q}}\left[e^{-(\gamma,\Delta S_{t+1})}|\mathcal{F}_t^S\right] \geq \\ \geq e^{-c_2}\mathsf{E}^{\mathsf{Q}}\left[e^{-(\gamma,\Delta S_{t+1})}1_{\{-(\gamma,\Delta S_{t+1})\geq c_3|\gamma|\}}|\mathcal{F}_t^S\right] \geq \\ \geq e^{c_3|\gamma|-c_2}\mathsf{Q}\left(-(\gamma,\Delta S_{t+1}) \geq c_3|\gamma|\big|\mathcal{F}_t^S\right).\end{aligned} \quad (61)$$

Inequality (60) follows from (61) and (58) (see Remark 9).

(2) From (59), Corollary 1, Theorem 2, properties of essential supremum and because $e^{-(\gamma,x)}$ is convex it follows that for any $t \in N_1$ the following inequality is true P-a.s.

$$\Phi(t,\gamma^\alpha,\omega) \leq \alpha\Phi(t,\gamma_1,\omega) + (1-\alpha)\Phi(t,\gamma_2,\omega), \quad (62)$$

where $\alpha \in [0,1]$, $\gamma^\alpha \triangleq \alpha\gamma_1 + (1-\alpha)\gamma_2$, $\gamma_1,\gamma_2 \in D_t$.

Obviously, $\Phi(t,\gamma,\omega)$ is $\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{F}_t^S$-measurable random variable and is finite P-a.s. for any $\gamma \in D_t$.

Let us denote: (i) $L_\infty^t \triangleq L_\infty^t\left(\mathbb{R}^{dt},\mathcal{B}\left(\mathbb{R}^{dt}\right),P\right)$ is normed space of P-a.s. finite random variables;

(ii) $\mathfrak{L}_t^\infty \triangleq \mathfrak{L}_t^\infty\left(\mathbb{R}^{dt},\mathcal{B}\left(\mathbb{R}^{dt}\right)\right)$ is normed space of measurable functions with uniform norm.

Suppose reflection $\rho_t : L_\infty^t \to \mathfrak{L}_t^\infty$ is lifting [15], i.e. if $\varphi_t \in L_\infty^t$ then:

(i) $\rho_t(\varphi_t) = \varphi_t$ P-a.s.;

(ii) if $\varphi_t(\omega) = 0$ P-a.s. then for any ω we have $\rho_t\big(\varphi_t(\omega)\big) = 0$;

(iii) $\rho\left(1_{\{\mathbb{R}^{dt}\}}(\omega)\right) = 1_{\{\mathbb{R}^{dt}\}}(\omega)$.

It is well known [15] that: (i) in this case lifting exists, (ii) $||\rho_t|| = 1$ and $\rho_t^2 = \rho_t$.

Obviously, for any $t \in N_1$ and $\gamma \in D_t$ we have $\Phi(t,\gamma,\omega) \in L_\infty^t$ and $\rho_t\left(\Phi(t,\gamma,\omega)\right) \in \mathfrak{L}_t^\infty$. So, from (60) it follows that for any (t,ω) function $\rho_t\left(\Phi(t,\gamma,\omega)\right)$ is convex non-negative function of γ. Consequently, for any (t,ω) it is continuous function of γ. The proof is complete.

3.3.4. PROOF OF THEOREM 3. From (58) (see Remark 9) and Lemma 2 it follows that:

(1) for each t and ω function $\Phi(t,\gamma,\omega)$ is a $\mathcal{B}\left(\mathbb{R}^d\right)$-measurable function of γ and

$$\lim_{|\gamma|\to\infty}\Phi(t,\gamma,\omega) = \infty \quad \mathsf{P}\text{-a.s.}; \quad (63)$$

(2) for each t and γ function $\Phi(t,\gamma,\omega)$ is $\mathcal{F}_t^S$-measurable.

Note that from definition of essential infimum, Theorems 1, 2 and definition of the sequence $\{\Phi(t,\gamma,\omega)\}_{t\in N_1}$ it follows that there is minimizing sequence $\{\gamma_{t+1}^{(k)}\}_{k\geq 1}$ such that P-a.s.

$$e^{-c_2} \leq \overline{V}_t = \operatorname*{ess\,inf}_{\gamma_{t+1}\in D_{t+1}} \operatorname*{ess\,sup}_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}} \left[\overline{V}_{t+1} e^{-(\gamma_{t+1},\Delta S_{t+1})} | \mathcal{F}_t^S\right] = \\ = \lim_{k\to\infty} \operatorname*{ess\,sup}_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}} \left[\overline{V}_{t+1} e^{-\left(\gamma_{t+1}^{(k)},\Delta S_{t+1}\right)} | \mathcal{F}_t^S\right] \leq e^{c_2}. \tag{64}$$

We will prove Theorem 3 by contradiction. According to (64) this means that there is no $\mathcal{F}_t^S$-measurable finite d-dimensional vector γ_{t+1}^* such that (9) is true. Consequently,

$$\lim_{k\to\infty} |\gamma_{t+1}^{(k)}| = \infty. \tag{65}$$

Taking into account (64) and (63) we get P-a.s.

$$e^{c_2} \geq \overline{V}_t = \lim_{k\to\infty} \operatorname*{ess\,sup}_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}} \left[\overline{V}_{t+1} e^{-\left(\gamma_{t+1}^{(k)},\Delta S_{t+1}\right)} | \mathcal{F}_t^S\right] = \\ = \lim_{k\to\infty} \Phi(t, \gamma_{t+1}^{(k)}, \omega) \geq \lim_{k\to\infty} c_4 e^{c_3|\gamma_{t+1}^{(k)}| - c_2} = \infty. \tag{66}$$

It is a contradiction, which means that our assumption (65) is wrong. That is why from $\{\gamma_{t+1}^{(k)}\}_{k\geq 1}$ it is possible to pick out converging subsequence $\{\gamma_{t+1}^{(k_l)}\}_{l\geq 1}$ such that

$$\gamma_{t+1}^* \triangleq \lim_{l\to\infty} \gamma_{t+1}^{(k_l)}. \tag{67}$$

Obviously, γ_{t+1}^* is a $\mathcal{F}_t^S$-measurable d-dimensional vector. Indeed, suppose $B \subseteq \mathbb{R}^d$ is an open sphere. For any t and ω it is true that $\Phi(t, \gamma, \omega)$ is continuous function of γ. So, we have

$$\{\gamma_t^* \in B\} = \bigcup_{q\in Q^d\cap B} \bigcap_{q'\in Q^d\setminus B} \{\Phi(t, q, \omega) < \Phi(t, q', \omega)\},$$

where Q^d is a space of d-dimensional vectors whose components are rational numbers.

Let us prove that $\gamma_{t+1}^* \in D_{t+1}$. From Corollary 1, Theorem 2, Lemma 2 and the Fatou Lemma it follows that P-a.s.

$$e^{c_2} \geq \overline{V}_t \geq \lim_{l\to\infty} \mathsf{E}^{\mathsf{Q}} \left[\overline{V}_{t+1} e^{-\left(\gamma_{t+1}^{(k_l)},\Delta S_{t+1}\right)} | \mathcal{F}_t^S\right] \geq \\ \geq \mathsf{E}^{\mathsf{Q}} \left[\overline{V}_{t+1} e^{-\left(\gamma_{t+1}^*,\Delta S_{t+1}\right)} | \mathcal{F}_t^S\right] \geq e^{-c_2} \mathsf{E}^{\mathsf{Q}} \left[e^{-\left(\gamma_{t+1}^*,\Delta S_{t+1}\right)} | \mathcal{F}_t^S\right].$$

From these inequalities it immediately follows that $e^{2c_2} \geq \operatorname*{ess\,sup}_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}} \left[e^{-\left(\gamma_{t+1}^*,\Delta S_{t+1}\right)} | \mathcal{F}_t^S\right]$. The proof is complete.

3.4. PROOF OF THEOREM 4. From Theorem 3 we obtain formula (10) for any $t \in N_1$ and $\mathsf{Q} \in \Re_N$. The measures Q and P are equivalent. Therefore, recalling the remarks of Subsection 3.1.1 and using Girsanov's theorem, we rewrite (10) as P-a.s.

$$\overline{V}_{t-1} \geq \mathsf{E}^{\mathsf{Q}} \left[\overline{V}_t e^{-(\gamma_t^*, \Delta S_t)} | \mathcal{F}_{t-1}^S \right] = \mathsf{E}^{\mathsf{P}} \left[\frac{z_t}{z_{t-1}} \overline{V}_t e^{-(\gamma_t^*, \Delta S_t)} | \mathcal{F}_{t-1}^S \right], \tag{68}$$

where $z_t = \dfrac{d\mathsf{Q}_t}{d\,\mathsf{P}_t}(\omega)$.

Let $\{g_t\}_{t \in N_1}$ be a sequence of random variables such that g_t is an arbitrary $\mathcal{F}_t^S$-measurable bounded random variable and $\left\{z_t, \mathcal{F}_t^S\right\}_{t \in N_0}$ admits the representation

$$z_t = z_{t-1} \frac{g_t}{\mathsf{E}^{\mathsf{P}} \left(g_t | \mathcal{F}_{t-1}^S \right)}, \quad z_t|_{t=0} = 1. \tag{69}$$

Obviously, $\{z_t\}_{t \in N_0} \in \overline{Z}_0^N$ and for any $t \in N_0$, the random variable z_t is $\mathcal{F}_t^S$-measurable and $0 < z_t < \infty$ P-a.s. Therefore, combining (68) with (69), we obtain

$$\overline{V}_{t-1} \mathsf{E}^{\mathsf{P}} \left[g_t | \mathcal{F}_{t-1}^S \right] \geq \mathsf{E}^{\mathsf{P}} \left[g_t \overline{V}_t e^{-(\gamma_t^*, \Delta S_t)} | \mathcal{F}_{t-1}^S \right] \ \mathsf{P}-\text{a.s.}$$

Hence, we have

$$0 \geq \mathsf{E}^{\mathsf{P}} \left[g_t \left(\frac{\overline{V}_t}{\overline{V}_{t-1}} e^{-(\gamma_t^*, \Delta S_t)} - 1 \right) | \mathcal{F}_{t-1}^S \right] \ \mathsf{P}-\text{a.s.} \tag{70}$$

Since

$$\exp\left\{ \Delta \ln \overline{V}_t - (\gamma_t^*, \Delta S_t) \right\} - 1 - \left[\Delta \ln \overline{V}_t - (\gamma_t^*, \Delta S_t) \right] \geq 0 \ \mathsf{P}-\text{a.s.},$$

from (70) we obtain the inequalities P-a.s.

$$0 \geq \mathsf{E}^{\mathsf{P}} \left\{ g_t \left[e^{\Delta \ln \overline{V}_t - (\gamma_t^*, \Delta S_t)} - 1 - \left(\Delta \ln \overline{V}_t - (\gamma_t^*, \Delta S_t) \right) + \Delta \ln \overline{V}_t - \right.\right.$$

$$\left.\left. - (\gamma_t^*, \Delta S_t) \right] | \mathcal{F}_{t-1}^S \right\} \geq \mathsf{E}^{\mathsf{P}} \left\{ g_t \left[\Delta \ln \overline{V}_t - (\gamma_t^*, \Delta S_t) \right] | \mathcal{F}_{t-1}^S \right\}.$$

Since g_t is arbitrary, we have P-a.s.

$$-\Delta C_t^* \triangleq \Delta \ln \overline{V}_t - (\gamma_t^*, \Delta S_t) \leq 0. \tag{71}$$

It follows from (71) that the $\mathcal{F}_t^S$-measurable random variable ΔC_t^* defined by (11) exists for any $t \in N_1$. It follows from (71) that for any $t \in N_1$, we have $\Delta C_t^* \geq 0$ P-a.s. Hence, $C_t^* \geq C_s^*$ P-a.s. for any $t \geq s$. It follows from (11) and our remark that P-a.s.

$$\begin{aligned} C_N^* &= \sum_{i=1}^{N} \Delta C_i^* = \sum_{i=1}^{N} \left[(\gamma_i^*, \Delta S_i) - \Delta \ln \overline{V}_i \right] = \\ &= \sum_{i=1}^{N} (\gamma_i^*, \Delta S_i) - \ln \overline{V}_N + \ln \overline{V}_0. \end{aligned} \tag{72}$$

As $\ln \overline{V}_N = f_N(S_\bullet)$ from the last equality we obtain (12). This completes the proof of the theorem.

3.5. P R O O F O F T H E O R E M 5. According to theorem's conditions ξ is P-a.s. bounded, so for any $\mathsf{Q} \in \mathfrak{R}_N$ expectation $\mathsf{E}^{\mathsf{Q}}\xi$ is also bounded, consequently, $\sup\limits_{\mathsf{Q}\in\mathfrak{R}_N} \mathsf{E}^{\mathsf{Q}}\xi$ is finite. Thus there is sequence $\{\mathsf{Q}_k\}_{k\geq 1}$ with $\mathsf{Q}_k \in \mathfrak{R}_N$ for any $k \geq 1$ such, that $\sup\limits_{\mathsf{Q}\in\mathfrak{R}_N} \mathsf{E}^{\mathsf{Q}}\xi = \lim\limits_{k\to\infty} \mathsf{E}^{\mathsf{Q}_k}\xi$.

For any $A \in \mathcal{F}$ let us define $\lambda(A) \triangleq \sum\limits_{i=1}^{\infty} \frac{\mathsf{Q}_k(A)}{2^i}$. Obviously, λ is a probability measure and $\mathsf{Q}_k \ll \lambda$ for any $k \geq 1$. Hence for any $k \geq 1$ there exists random variable $0 \leq X_k \triangleq \frac{d\mathsf{Q}_k}{d\lambda}$ called density of Q_k with respect to λ, $\mathsf{E}^{\lambda} X_k = 1$ [18]. Then $\mathsf{E}^{\mathsf{Q}_k}\xi = \mathsf{E}^{\lambda} X_k \xi$, $k \geq 1$.

Suppose $\{X_k\}_{k\geq 1}$ is weakly relatively compact in $L^1(\Omega, \mathcal{F}, \lambda)$. In that case the theorem of Eberlein-Smulian [3] guarantees existence of subsequence $\{X_{k_m}\}_{m\geq 1}$ with weak limit in topology of $L^1(\Omega, \mathcal{F}, \lambda)$, i.e. there exists $X \in L^1(\Omega, \mathcal{F}, \lambda)$ such, that for any $Y \in L^{\infty}(\Omega, \mathcal{F}, \lambda)$:

$$\lim_{m\to\infty} \mathsf{E}^{\lambda} X_{k_m} Y = \mathsf{E}^{\lambda} XY. \tag{73}$$

We define $\mathsf{Q}^*(A) \triangleq \mathsf{E}^{\lambda} X 1_{\{A\}}$, $A \in \mathcal{F}$. Equality (73) holds true for P-a.s. bounded ξ, that is why

$$\lim_{m\to\infty} \mathsf{E}^{\lambda} X_{k_m} \xi = \mathsf{E}^{\lambda} X \xi = \mathsf{E}^{\mathsf{Q}^*}\xi = \lim_{m\to\infty} \mathsf{E}^{\mathsf{Q}_{k_m}}\xi. \tag{74}$$

As $\left\{\mathsf{E}^{\mathsf{Q}_k}\xi\right\}_{k\geq 1}$ has a limit, so $\left\{\mathsf{E}^{\mathsf{Q}_{k_m}}\xi\right\}_{m\geq 1}$ has the same limit. Thus we have

$$\sup_{\mathsf{Q}\in\mathfrak{R}_N} \mathsf{E}^{\mathsf{Q}}\xi = \lim_{k\to\infty} \mathsf{E}^{\mathsf{Q}_k}\xi = \lim_{m\to\infty} \mathsf{E}^{\mathsf{Q}_{k_m}}\xi = \mathsf{E}^{\mathsf{Q}^*}\xi. \tag{75}$$

The proof is complete.

3.6. In this subsection, we prove Theorem 6.

3.6.1. To prove Theorem 6, we need the following lemma.

Lemma 3 *Suppose that the assumptions of Theorem 4 are satisfied. Then the following assertions are true:*

(1)

$$0 \leq \exp\left\{ f_N\left(S_\bullet\right) - \sum_{i=1}^{N}\left(\gamma_i^*, \Delta S_i\right)\right\} \leq e^{c_2} \quad \mathsf{P}-a.s.; \tag{76}$$

(2) for any $t \in N_1$,

$$0 \leq \overline{V}_t e^{-(\gamma_t^*, \Delta S_t)} \leq e^{c_2} \quad \mathsf{P}-a.s.; \tag{77}$$

PROOF OF LEMMA 3. (1) We first prove (76). From (12) we obtain

$$\overline{V}_0 e^{-C_N^*} = \exp\left\{ f_N\left(S_\bullet\right) - \sum_{i=1}^{N}\left(\gamma_i^*, \Delta S_i\right)\right\} \quad \mathsf{Q}-\text{a.s.} \tag{78}$$

As $e^{-c_2} \leq \overline{V}_0 \leq e^{c_2}$ and $C_N^* \geq 0$ Q-a.s., (76) follows from (78).

(2) Let us prove inequality (77). The left inequality in (77) is obvious. So we have to prove right inequality in (77). From (11) it follows that $\frac{\overline{V}_t}{\overline{V}_{t-1}} e^{-(\gamma_t^*, \Delta S_t) + \Delta C_t^*} = 1$ Q-a.s. for any $t \in N_1$ and $\mathsf{Q} \in \Re_N$. We know that $\Delta C_t^* \geq 0$ Q-a.s. So, from the last equality it follows that $\overline{V}_t e^{-(\gamma_t^*, \Delta S_t)} \leq \overline{V}_{t-1}$ Q-a.s. That is why from Theorem 2 the right inequality in (77) follows. The proof is complete.

3.6.2. PROOF OF THEOREM 6. Now let us prove equality (14). Let

$$\xi_t\left(\omega\right) = \eta_{t-1}\overline{V}_t e^{-(\gamma_t^*, \Delta S_t)},$$

where η_{t-1} is an $\mathcal{F}_{t-1}^S$-measurable bounded random variable, $\left(\overline{V}_t, \mathcal{F}_t^S\right)_{t\in N_0}$ satisfies recurrent relation (5) and $\left(\gamma_t^*, \mathcal{F}_{t-1}^S\right)_{t\in N_1}$ is defined by (9). It follows from the second assertion of Lemma 3 that for any $t \in N_1$, the $\overline{V}_t$ and $\overline{V}_t e^{-(\gamma_t^*, \Delta S_t)}$ are $\mathcal{F}_t^S$ measurable bounded random variables. Consider $\sup\limits_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}} \eta_{t-1}\overline{V}_t e^{-(\gamma_t^*, \Delta S_t)}$. On the one hand, the properties of the essential supremum and the conditional expectation $\mathsf{E}^{\mathsf{Q}}\left[\bullet|\mathcal{F}_{t-1}^S\right]$ and also Theorem 3 imply that

$$\sup_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}}\eta_{t-1}\overline{V}_t e^{-(\gamma_t^*, \Delta S_t)} = \sup_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}}\eta_{t-1}\mathsf{E}^{\mathsf{Q}}\left[\overline{V}_t e^{-(\gamma_t^*, \Delta S_t)}|\mathcal{F}_{t-1}^S\right] =$$

$$= \sup_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}}\eta_{t-1} \operatorname*{ess\,sup}_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}}\left[\overline{V}_t e^{-(\gamma_t^*, \Delta S_t)}|\mathcal{F}_{t-1}^S\right] = \sup_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}}\eta_{t-1}\overline{V}_{t-1}. \tag{79}$$

Further, taking (75) into account, we obtain the following equalities

$$\sup_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}}\eta_{t-1}\overline{V}_{t-1} = \mathsf{E}^{\mathsf{Q}^*}\eta_{t-1}\overline{V}_{t-1}, \tag{80}$$

$$\sup_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}}\eta_{t-1} \operatorname*{ess\,sup}_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}}\left[\overline{V}_t e^{-(\gamma_t^*,\Delta S_t)}|\mathcal{F}_{t-1}^S\right] = \tag{81}$$
$$= \mathsf{E}^{\mathsf{Q}^*}\left\{\eta_{t-1} \operatorname*{ess\,sup}_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}}\left[\overline{V}_t e^{-(\gamma_t^*,\Delta S_t)}|\mathcal{F}_{t-1}^S\right]\right\}.$$

Since the random variable η_{t-1} is arbitrary, it follows from (79), (80), and (81) that for any $t\in N_1$,

$$\overline{V}_{t-1} = \operatorname*{ess\,sup}_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}}\left[\overline{V}_t e^{-(\gamma_t^*,\Delta S_t)}|\mathcal{F}_{t-1}^S\right] \quad \mathsf{Q}^* - \text{a.s.} \tag{82}$$

On the other hand, (74)–(75) and the properties of the conditional expectation imply the equalities

$$\sup_{\mathsf{Q}\in\Re_N} \mathsf{E}^{\mathsf{Q}}\eta_{t-1}\overline{V}_t e^{-(\gamma_t^*,\Delta S_t)} = \lim_{n\to\infty} \mathsf{E}^{\mathsf{Q}^{(n)}}\eta_{t-1}\overline{V}_t e^{-(\gamma_t^*,\Delta S_t)} =$$

$$= \lim_{n\to\infty} \mathsf{E}^{\lambda}\eta_{t-1}\overline{V}_t e^{-(\gamma_t^*,\Delta S_t)}\frac{d\mathsf{Q}^{(n)}}{d\lambda}(\omega) =$$
$$= \mathsf{E}^{\lambda}\eta_{t-1}\overline{V}_t e^{-(\gamma_t^*,\Delta S_t)}\chi(\omega) = \mathsf{E}^{\mathsf{Q}^*}\eta_{t-1}\overline{V}_t e^{-(\gamma_t^*,\Delta S_t)} = \tag{83}$$
$$= \mathsf{E}^{\mathsf{Q}^*}\eta_{t-1}\mathsf{E}^{\mathsf{Q}^*}\left[\overline{V}_t e^{-(\gamma_t^*,\Delta S_t)}|\mathcal{F}_{t-1}^S\right].$$

Using (79), (80) and (83), we obtain

$$\mathsf{E}^{\mathsf{Q}^*}\eta_{t-1}\overline{V}_{t-1} = \mathsf{E}^{\mathsf{Q}^*}\eta_{t-1}\mathsf{E}^{\mathsf{Q}^*}\left[\overline{V}_t e^{-(\gamma_t^*,\Delta S_t)}|\mathcal{F}_{t-1}^S\right]. \tag{84}$$

As η_{t-1} is arbitrary, taking (84) into account, we obtain, for any $t\in N_1$,

$$\overline{V}_{t-1} = \mathsf{E}^{\mathsf{Q}^*}\left[\overline{V}_t e^{-(\gamma_t^*,\Delta S_t)}|\mathcal{F}_{t-1}^S\right] \quad \mathsf{Q}^* - \text{a.s.} \tag{85}$$

Obviously, $\overline{V}_t|_{t=N} = \exp\{f_N(S_\bullet)\}$. This and recurrent relation (85) imply equality for any $t\in N_1$

$$\overline{V}_t = \mathsf{E}^{\mathsf{Q}^*}\left[\exp\left\{f_N(S_\bullet) - \sum_{i=t+1}^{N}(\gamma_i^*,\Delta S_i)\right\}|\mathcal{F}_t^S\right] = I_t^{\mathsf{Q}^*,\gamma_{t+1}^{*N}}(S_0^t) \quad \mathsf{Q}^* - \text{a.s.}$$

Thus, equality (3) is proved. So, there are the worst-case measure Q^* and the minimax strategy $\{\gamma_t^*\}_{t\in N_1}$, i.e. there exists the minimax bistrategy $(\mathsf{Q}^*,\gamma_1^{*N})$. The proof is complete.

3.7. Proof of Corollary 2. For convenience of presentation let us denote $G_t \triangleq \left\{\omega\in\Omega : \triangle\ln\overline{V}_t(\omega) = (\gamma_t^*,\triangle S_t)(\omega) - \triangle C_t^*(\omega)\right\}$. Obviously, G_t is a $\mathcal{F}_t^S$-measurable set. From the proof of Theorem 5 it follows that there are sequence $\{\mathsf{Q}^{(n)}\}_{n\geq 1}$, $\mathsf{Q}^{(n)}\in\Re_N$, and probability measure Q^* such that for any $A\in\mathcal{F}_N^S$ we have $\mathsf{Q}^*(A) = \lim_{n\to\infty}\mathsf{Q}^{(n)}(A)$. According to Theorem 4

$\mathsf{Q}^{(n)}(G_t) = 1$ for any $n \geq 1$. Hence $\mathsf{Q}^*(G_t) = \lim_{n\to\infty} \mathsf{Q}^{(n)}(G_t) = 1$. The proof is complete.

3.8. PROOF OF THEOREM 7. (1) Suppose that assertion (1) holds. Let us prove assertion (2). Let Q^* be the worst-case probability distribution. We must prove equality (14). Let us assume the converse, i.e., suppose that there is $t \in N_1$ such that the following inequality holds:

$$\mathsf{Q}^* \left\{ \overline{V}_{t-1} > \mathsf{E}^{\mathsf{Q}^*} \left[\overline{V}_t e^{-(\gamma_t^*, \Delta S_t)} | \mathcal{F}_{t-1}^S \right] \right\} > 0.$$

Therefore, $\overline{V}_t > I_t^{\mathsf{Q}^*, \gamma_{t+1}^{*N}} (S_0^t)$. This means that Q^* is not the worst-case measure. This contradiction proves the assertion.

(2) Suppose that assertion (2) holds. Let us prove assertion (3). Let us multiply both sides of (14) by $\exp\left\{ -\sum_{i=1}^{t-1} (\gamma_i^*, \Delta S_i) \right\}$. Taking the definition of the sequence $\left\{ \overline{\mu}_t, \mathcal{F}_t^S \right\}_{t\in N_0}$ into account and using the properties of the conditional expectations, we conclude that

$$\overline{\mu}_{t-1} = \overline{V}_{t-1} \exp\left\{ -\sum_{i=1}^{t-1} (\gamma_i^*, \Delta S_i) \right\} =$$

$$= \mathsf{E}^{\mathsf{Q}^*} \left[\overline{V}_t \exp\left\{ -\sum_{i=1}^{t} (\gamma_i^*, \Delta S_i) \right\} | \mathcal{F}_{t-1}^S \right] = \mathsf{E}^{\mathsf{Q}^*} \left[\overline{\mu}_t | \mathcal{F}_{t-1}^S \right] \quad \mathsf{Q} - \text{a.s.}$$

It follows from Lemma 3 and Theorem 3 that $\mathsf{E}^{\mathsf{Q}^*} \overline{\mu}_t < \infty$. Hence, the sequence $\left\{ \overline{\mu}_t, \mathcal{F}_t^S \right\}_{t\in N_0}$ is a martingale with respect to Q^*.

(3) Suppose that (3) holds. Let us prove (1). It follows from the definition of the S-estimating sequence $\left\{ \overline{\mu}_t, \mathcal{F}_t^S \right\}_{t\in N_0}$ and the assumptions of the theorem that Q^*-a.s.

(a) $\overline{\mu}_N = \exp\left\{ f_N(S_\bullet) - \sum_{i=1}^{N} (\gamma_i^*, \Delta S_i) \right\}$;

(b) $\overline{\mu}_0 = \overline{V}_0 = \operatorname*{ess\,inf}_{\gamma_1^N \in D_1^N} \operatorname*{ess\,sup}_{\mathsf{Q}\in\Re_N} I_0^{\mathsf{Q}, \gamma_1^N} (S_0)$;

(c) $\left\{ \overline{\mu}_t, \mathcal{F}_t^S \right\}_{t\in N_0}$ is a martingale with respect to Q^*.

Therefore, we have (14) which implies

$$\overline{V}_t = I_t^{\mathsf{Q}^*, \gamma_{t+1}^{*N}} \left(S_0^t \right) \quad \mathsf{Q}^* - \text{a.s.} \tag{86}$$

By Remark 4 for any $t \in N_0$ and for any measure $\mathsf{Q} \in \Re_N$, we have

$$\overline{\mu}_t \geq \mathsf{E}^{\mathsf{Q}} \left[\overline{\mu}_{t+1} | \mathcal{F}_t^S \right] \quad \mathsf{Q} - \text{a.s.} \tag{87}$$

Hence, it follows from (10), (86), (87), Remark 1 and recurrent relation (38) that for any $t \in N_0$ and $\mathsf{Q} \in \Re_N$, we have

$$I_t^{\mathsf{Q}^*, \gamma_{t+1}^{*N}}\left(S_0^t\right) = \overline{V}_t \geq \mathsf{E}^{\mathsf{Q}}\left[\overline{V}_{t+1} e^{-\left(\gamma_{t+1}^*, \Delta S_{t+1}\right)} | \mathcal{F}_t^S\right] \geq$$

$$\geq \mathsf{E}^{\mathsf{Q}}\left[I_{t+1}^{\mathsf{Q}, \gamma_{t+2}^{*N}}\left(S_0^{t+1}\right) e^{-\left(\gamma_{t+1}^*, \Delta S_{t+1}\right)} | \mathcal{F}_t^S\right] = I_t^{\mathsf{Q}, \gamma_{t+1}^{*N}}\left(S_0^t\right) \quad \mathsf{Q}^* - \text{a.s.}$$

Thus, the Q^* is the worst-case measure. This completes the proof of Theorem 7.

3.9. Proof of Theorem 8. The proof follows from the Assertions 1–7.

§4. Proofs of Theorems 13–20

4.1. In this subsection, we prove Theorem 10, which establishes the relationship between problem (2) and the problem of an European option pricing in an incomplete $\left(1, S^{(1)}, \ldots, S^{(d)}\right)$-market.

4.1.1. Proof of Theorem 10. It follows from Theorem 3 that there is a strategy $\{\gamma_t^*\}_{t \in N_1} \in D_1^N$ satisfying (9). Therefore, by (18), for each $t \in N_1$, there is a predictable sequence $\{\beta_t^*\}_{t \in N_0}$ such that

$$\Delta \beta_t^* = -\left(S_{t-1}, \Delta \gamma_t^*\right), \quad \beta_t^*|_{t=0} = \beta_0^*. \tag{88}$$

The value of β_0^* will be found later. So, we have just constructed a self-financing portfolio $\pi^* = (\beta_t^*, \gamma_t^*)_{t \in N_0}$. Therefore, according to (17), for any $t \in N_0$, the capital $X_t^{\pi^*}$ of the portfolio π^* is defined by the formula

$$X_t^{\pi^*} = \beta_t^* + \left(\gamma_t^*, S_t\right). \tag{89}$$

Hence, for any $t \in N_1$, the following equality holds Q-a.s.:

$$\Delta X_t^{\pi^*} \triangleq X_t^{\pi^*} - X_{t-1}^{\pi^*} = \Delta \beta_t^* + \Delta\left(\gamma_t^*, S_t\right). \tag{90}$$

Combining (90) with (88), we obtain

$$\Delta X_t^{\pi^*} \triangleq \left(\gamma_t^*, \Delta S_t\right) \quad \mathsf{Q} - \text{a.s.} \tag{91}$$

Theorem 4 (see (11)) yields that for each $t \in N_1$,

$$\left(\gamma_t^*, \Delta S_t\right) = \Delta \ln \overline{V}_t + \Delta C_t^* \ \ \mathsf{Q} - \text{a.s.}, \tag{92}$$

where $\left(C_t^*, \mathcal{F}_t\right)_{t \in N_0}$ is such that

(i) $C_0^* = 0$;

(ii) for each $t \in N_0$ and $\mathsf{Q} \in \Re_N$, the inequality $\Delta C_t^* \geq 0$ holds Q-a.s. Combining (91) with (92), we obtain

$$\Delta \left(X_t^{\pi^*} - \ln \overline{V}_t - C_t^* \right) = 0 \ \ \mathsf{Q} - \text{a.s.}$$

The last equality yields for any $t \in N_0$,

$$X_t^{\pi^*} - \ln \overline{V}_t - C_t^* = X_0^{\pi^*} - \ln \overline{V}_0 - C_0^* \ \ \mathsf{Q} - \text{a.s.} \tag{93}$$

Let

$$X_0^{\pi^*} = \ln \overline{V}_0 \ \ \mathsf{Q} - \text{a.s.} \tag{94}$$

Then, it follows from (93), (94) and equality $C_0^* = 0$ that for any $t \in N_0$ and $\mathsf{Q} \in \Re_N$,

$$X_t^{\pi^*} - C_t^* = \ln \overline{V}_t \ \ \mathsf{Q} - \text{a.s.} \tag{95}$$

Since $\left(C_t^*, \mathcal{F}_t^S \right)_{t \in N_0}$ is a nondecreasing sequence such that $C_0^* = 0$, it follows from (19) that $\widehat{X}_t^{\pi^*} = X_t^{\pi^*} - C_t^*$ is the capital of the self-financing portfolio π^* with consumption C_t^* at time $t \in N_0$. Also, from (95) we conclude that $\widehat{X}_t^{\pi^*} = \ln \overline{V}_t$ is the capital of the self-financing portfolio with consumption (π^*, C^*). As $X_0^{\pi^*} = \ln \overline{V}_0$, without loss of generality, we can assume that $\beta_0^* = \ln \overline{V}_0$ and $\gamma_0^* = 0$.

It follows from Theorem 4 (see (11)) that $\widehat{X}_N^{\pi^*} = \ln \overline{V}_N = f_N \left(S_\bullet \right)$ Q-a.s. with respect to any measure $\mathsf{Q} \in \Re_N$. Hence, we have

$$f_N \left(S_\bullet \right) = \ln \overline{V}_N = \ln \overline{V}_0 + \sum_{t=1}^{N} \left(\gamma_t^*, \Delta S_t \right) - C_N^* \ \ \mathsf{Q} - \text{a.s.}$$

So, the self-financing portfolio with consumption (π^*, C^*) is a perfect superhedging portfolio with consumption.

It remains to prove that (π^*, C^*) is the minimal perfect superhedging portfolio with consumption. To prove this, we need the following lemma.

4.1.2.

Lemma 4 *Let $f_N \left(S_\bullet \right)$ be a bounded $\mathcal{F}_N^S$-measurable contingent claim, and let (π^*, C^*) be the perfect superhedging portfolio with consumption defined by (9), (11), and (18). We assume that (π, C) is any other perfect superhedging portfolio with consumption, i.e., $(\pi, C) \neq (\pi^*, C^*)$. Then for any $t \in N_0$ and $\mathsf{Q} \in \Re_N$, the following inequality holds Q-a.s.:*

$$1 \geq \exp \left\{ \widehat{X}_t^{\pi^*} - \widehat{X}_t^{\pi} \right\}. \tag{96}$$

P R O O F O F L E M M A 4. It follows from Theorem 4 and the assumptions of the lemma that the contingent claim admits the following representations with respect to any measure $\mathsf{Q} \in \Re_N$:

$$\begin{aligned} f_N\left(S_\bullet\right) &= \widehat{X}_{t_0}^{\pi^*} + \sum_{i=t_0+1}^{N}\left(\gamma_i^*, \Delta S_i\right) - \left(C_N^* - C_{t_0}^*\right) = \\ &= \widehat{X}_{t_0}^{\pi} + \sum_{i=t_0+1}^{N}\left(\gamma_i, \Delta S_i\right) - \left(C_N - C_{t_0}\right) \quad \mathsf{Q} - \text{a.s.}, \end{aligned}$$

where $t_0 \in N_0$ is arbitrary. Hence, we have the following equality with respect to any measure $\mathsf{Q} \in \Re_N$:

$$\begin{aligned} \widehat{X}_{t_0}^{\pi^*} - \widehat{X}_{t_0}^{\pi} - \sum_{i=t_0+1}^{N} \Delta C_i^* = \sum_{i=t_0+1}^{N}\left(\gamma_i - \gamma_i^*, \Delta S_i\right) - \\ - \left(C_N - C_{t_0}\right) \quad \mathsf{Q} - \text{a.s.} \end{aligned} \tag{97}$$

Since $C_N - C_{t_0} \geq 0$ Q-a.s., it follows from (97) that

$$\widehat{X}_{t_0}^{\pi^*} - \widehat{X}_{t_0}^{\pi} - \sum_{i=t_0+1}^{N} \Delta C_i^* \leq \sum_{i=t_0+1}^{N}\left(\gamma_i - \gamma_i^*, \Delta S_i\right) \quad \mathsf{Q} - \text{a.s.} \tag{98}$$

For any $t \in \{t_0+1, \ldots, N\}$ the capital of the perfect superhedging portfolio with consumption (π^*, C^*) allows the representation $\widehat{X}_t^{\pi *} = \ln \overline{V}_t$ Q-a.s. So, use of (14) obtains

$$\Delta \widehat{X}_t^{\pi^*} = \left(\gamma_t^*, \Delta S_t\right) - \Delta C_t^* \quad \mathsf{Q} - \text{a.s.}$$

Therefore, combining (98) with the last equality, we have

$$\widehat{X}_{t_0}^{\pi^*} - \widehat{X}_{t_0}^{\pi} + \sum_{i=t_0+1}^{N}\left[\Delta \widehat{X}_i^{\pi^*} - \left(\gamma_i, \Delta S_i\right)\right] \leq 0 \quad \mathsf{Q} - \text{a.s.}$$

Hence, it obviously follows that

$$\exp\left\{\widehat{X}_{t_0}^{\pi^*} - \widehat{X}_{t_0}^{\pi} + \sum_{i=t_0+1}^{N}\left[\Delta \widehat{X}_i^{\pi^*} - \left(\gamma_i, \Delta S_i\right)\right]\right\} \leq 1 \quad \mathsf{Q} - \text{a.s.} \tag{99}$$

Now let us calculate the conditional expectation $\mathsf{E}^{\mathsf{Q}}\left[\bullet | \mathcal{F}_{t_0}^S\right]$ of the random variables at both sides of (99) with respect to any $\mathsf{Q} \in \Re_N$. Recall that

$\widehat{X}_{t_0}^{\pi^*} = \ln \overline{V}_{t_0}$, $\widehat{X}_N^{\pi^*} = f_N(S_\bullet)$. Using (1), we obtain the inequality Q-a.s.

$$\begin{aligned} 1 &\geq \exp\left\{\widehat{X}_{t_0}^{\pi^*} - \widehat{X}_{t_0}^{\pi}\right\} \mathsf{E}^{\mathsf{Q}} \left[\exp\left\{\widehat{X}_N^{\pi^*} - \ln \overline{V}_{t_0} - \sum_{i=t_0+1}^{N} (\gamma_i, \Delta S_i)\right\} | \mathcal{F}_{t_0}^S\right] = \\ &= \frac{\exp\left\{\widehat{X}_{t_0}^{\pi^*} - \widehat{X}_{t_0}^{\pi}\right\}}{\overline{V}_{t_0}} \mathsf{E}^{\mathsf{Q}} \left[\exp\left\{f_N(S_\bullet) - \sum_{i=t_0+1}^{N} (\gamma_i, \Delta S_i)\right\} | \mathcal{F}_{t_0}^S\right] = \\ &= \exp\left\{\widehat{X}_{t_0}^{\pi^*} - \widehat{X}_{t_0}^{\pi}\right\} \frac{I_{t_0}^{\mathsf{Q}, \gamma_{t_0+1}^N}\left(S_0^{t_0}\right)}{\overline{V}_{t_0}}. \end{aligned}$$

As the left-hand side of the last inequality does not depend on $\mathsf{Q} \in \Re_N$, for each $\gamma_{t_0+1}^N \in D_{t_0+1}^N$, we obtain the inequality

$$1 \geq \exp\left\{\widehat{X}_{t_0}^{\pi^*} - \widehat{X}_{t_0}^{\pi}\right\} \frac{\operatorname*{ess\,sup}_{\mathsf{Q} \in \Re_N} I_{t_0}^{\mathsf{Q}, \gamma_{t_0+1}^N}\left(S_0^{t_0}\right)}{\overline{V}_{t_0}} \quad \mathsf{Q} - \text{a.s.} \tag{100}$$

In turn, inequality (100) implies that for any $t \in N_0$ and $\mathsf{Q} \in \Re_N$, the following inequality holds Q-a.s.:

$$1 \geq \exp\left\{\widehat{X}_{t_0}^{\pi^*} - \widehat{X}_{t_0}^{\pi}\right\} \frac{\operatorname*{ess\,inf}_{\gamma_{t_0+1}^N \in D_{t_0+1}^N} \operatorname*{ess\,sup}_{\mathsf{Q} \in \Re_N} I_{t_0}^{\mathsf{Q}, \gamma_{t_0+1}^N}\left(S_0^{t_0}\right)}{\overline{V}_{t_0}} = \exp\left\{\widehat{X}_{t_0}^{\pi^*} - \widehat{X}_{t_0}^{\pi}\right\}.$$

This completes the proof of the lemma.

4.1.3. Here we complete the proof of Theorem 10. Now let us prove that a perfect superhedging portfolio with consumption (π^*, C^*) is the minimal one. We assume the opposite, i.e., we assume that there is a moment $t_0 \in N_0$, a measure $\mathsf{Q} \in \Re_N$, and a perfect superhedging portfolio with consumption (π, C) such that $\mathsf{Q}\left(\widehat{X}_{t_0}^{(\pi^*)} > \widehat{X}_{t_0}^{(\pi)}\right) > 0$. On the other hand, (96) implies that $\mathsf{Q}\left(\widehat{X}_{t_0}^{(\pi^*)} > \widehat{X}_{t_0}^{(\pi)}\right) = 0$. This contradiction proves that the perfect superhedging portfolio with consumption (π^*, C^*) is the minimal one. The theorem is proved.

4.2. P R O O F O F T H E O R E M 11. (1) It follows from Corollary 1 and Theorem 6 that for any $\gamma \in D_t$, the following inequality holds Q^*-a.s.:

$$\begin{aligned} \overline{V}_{t-1} &\leq \operatorname*{ess\,sup}_{\mathsf{Q} \in \Re_N} \mathsf{E}^{\mathsf{Q}} \left[\overline{V}_t e^{-(\gamma, \Delta S_t)} | \mathcal{F}_{t-1}^S\right] = \\ &= \mathsf{E}^{\mathsf{Q}^*} \left[\overline{V}_t e^{-(\gamma, \Delta S_t)} | \mathcal{F}_{t-1}^S\right]. \end{aligned} \tag{101}$$

Suppose that $\gamma = \gamma_t^* + h\overline{\gamma}_t$, where $h \in (0,1]$ is arbitrary and $\overline{\gamma}_t$ is an $\mathcal{F}_{t-1}^S$-measurable vector. Without loss of generality, we can assume that $|\overline{\gamma}_t| \leq 1$. Then (101) yields the inequality Q^*-a.s.

$$\begin{aligned} \overline{V}_{t-1} &\leq \mathsf{E}^{\mathsf{Q}^*}\left[\overline{V}_t e^{-(\gamma_t^*,\Delta S_t)} e^{-h(\overline{\gamma}_t,\Delta S_t)}|\mathcal{F}_{t-1}^S\right] = \\ &= \overline{V}_{t-1}\mathsf{E}^{\mathsf{Q}^*}\left[\exp\left\{\Delta \ln \overline{V}_t - (\gamma_t^*,\Delta S_t)\right\} e^{-h(\overline{\gamma}_t,\Delta S_t)}|\mathcal{F}_{t-1}^S\right]. \end{aligned} \tag{102}$$

It follows from Corollary 2 that

$$\Delta \ln \overline{V}_t - (\gamma_t^*,\Delta S_t) = -\Delta C_t^* \leq 0 \;\; \mathsf{Q}^* - \text{a.s.}.$$

Therefore, (102) can be sharpened as follows Q^*-a.s.

$$\begin{aligned} 1 &\leq \mathsf{E}^{\mathsf{Q}^*}\left[\exp\left\{-\Delta C_t^* - h\left(\overline{\gamma}_t,\Delta S_t\right)\right\}|\mathcal{F}_{t-1}^S\right] \leq \\ &\leq \mathsf{E}^{\mathsf{Q}^*}\left[e^{-h(\overline{\gamma}_t,\Delta S_t)}|\mathcal{F}_{t-1}^S\right]. \end{aligned} \tag{103}$$

Using the Newton–Leibniz formula, we can rewrite (103) for any $h \in (0,1]$ as

$$\begin{aligned} 0 &\leq \mathsf{E}^{\mathsf{Q}^*}\left[\frac{1}{h}\int_0^h \frac{d}{du} e^{-u(\overline{\gamma}_t,\Delta S_t)} du|\mathcal{F}_{t-1}^S\right] = \\ &= -\mathsf{E}^{\mathsf{Q}^*}\left[(\overline{\gamma}_t,\Delta S_t)\frac{1}{h}\int_0^h e^{-u(\overline{\gamma}_t,\Delta S_t)} du|\mathcal{F}_{t-1}^S\right] \;\; \mathsf{Q}^* - \text{a.s.} \end{aligned} \tag{104}$$

Passing to the limit as $h \to 0$ and using Fatou's lemma, we obtain Q^*-a.s.

$$\begin{aligned} 0 &\geq \lim_{h\downarrow 0}\mathsf{E}^{\mathsf{Q}^*}\left[(\overline{\gamma}_t,\Delta S_t)\frac{1}{h}\int_0^h e^{-u(\overline{\gamma}_t,\Delta S_t)} du|\mathcal{F}_{t-1}^S\right] \geq \\ &\geq \mathsf{E}^{\mathsf{Q}^*}\left[(\overline{\gamma}_t,\Delta S_t)\lim_{h\downarrow 0}\frac{1}{h}\int_0^h e^{-u(\overline{\gamma}_t,\Delta S_t)} du|\mathcal{F}_{t-1}^S\right] = \mathsf{E}^{\mathsf{Q}^*}\left[(\overline{\gamma}_t,\Delta S_t)|\mathcal{F}_{t-1}^S\right]. \end{aligned}$$

Since $\overline{\gamma}_t$ is arbitrary, we obtain

$$\mathsf{E}^{\mathsf{Q}^*}\left[\Delta S_t|\mathcal{F}_{t-1}^S\right] = 0 \;\; \mathsf{Q}^* - \text{a.s.}$$

Therefore, the sequence $\{S_t,\mathcal{F}_t\}_{t\in N_0}$ is a local martingale with respect to the measure Q^*. Thus, Q^* is a martingale measure. This completes the proof of the theorem.

4.3. PROOF OF THEOREM 12. On the one hand, Corollary 2 implies that for any $t \in N_1$, the probability $\mathsf{Q}^* \{\Delta C_t^* \geq 0\} = 1$. Therefore, for any $t \in N_1$, we have

$$1 - e^{-\Delta C_t^*} \geq 0 \;\; \mathsf{Q}^* - \text{a.s.} \tag{105}$$

On the other hand, Theorem 6 and (14) imply that for any $t \in N_1$, we have

$$\mathsf{E}^{\mathsf{Q}^*}\left[1 - e^{-\Delta C_t^*} | \mathcal{F}_{t-1}^S\right] = 0 \;\; \mathsf{Q}^* - \text{a.s.} \tag{106}$$

Combining (105) and (106), we obtain that for any $t \in N_1$,

$$\Delta C_t^* = 0 \;\; \mathsf{Q}^* - \text{a.s.} \tag{107}$$

Since $C_0^* = 0$, equality (107) implies that for any $t \in N_0$, the probability $\mathsf{Q}^* \{C_t^* = 0\} = 1$.

Let us prove (30). From Theorem 4 (see(11)) and (107) it follows that for any $t \in N_1$, we have Q^*-a.s.

$$\Delta \ln \overline{V}_t = (\gamma_t^*, \Delta S_t)\,.$$

We sum up last equalities gives Q^*-a.s. for all $t = 0, \ldots, k \leq N$:

$$\overline{V}_k = \ln \overline{V}_0 + \sum_{i=1}^{k} (\gamma_i^*, \Delta S_i) \tag{108}$$

In particular, as $\ln \overline{V}_t|_{t=N} = f_N(S_\bullet)$, we have Q^*-a.s.

$$\overline{V}_t|_{t=N} = f_N(S_\bullet) = \ln \overline{V}_0 + \sum_{i=1}^{N} (\gamma_i^*, \Delta S_i) \tag{109}$$

Let us calculate the conditional expectation $\mathsf{E}^{\mathsf{Q}^*}\left[\bullet | \mathcal{F}_0^S\right]$ for both sides of (109). The measure Q^* is a martingale measure. Hence we have

$$\ln \overline{V}_0 = \mathsf{E}^{\mathsf{Q}^*}\left[f_N(S_\bullet) | \mathcal{F}_0^S\right].$$

This and (109) imply (30). This completes the proof.

Remark 10 *Suppose martingale $\{\overline{V}_t, \mathcal{F}_t^S\}_{t \in N_0}$ admits decomposition (108) with respect to measure Q^* belonging to closure of a set $\mathfrak{R}_N$ (in the topology of weak convergence for probability measures). It easy to see, that triplet $\left(\mathsf{Q}^*, \gamma_1^{*N}, \overline{V}_0\right)$ is a solution of problem (2). Indeed, (109) is equal to: $\ln \overline{V}_0 = f_N - \sum\limits_{i=1}^{N} (\gamma_i^*, \Delta S_i) \;\; \mathsf{Q}^* - a.s.$ Take exponent and, after that, conditional expectation $\mathsf{E}^{\mathsf{Q}^*}\left[\cdot | \mathcal{F}_0^S\right]$. So, we have (3) from the definition of the solution for the problem (2).*

4.4.

Remark 11 *Let us consider* $(\Omega, \mathcal{F}, \mathsf{P})$ *and the set of equivalent (to* P*) probability measures* $\Re$. *Then Dirak measure for any* $\hat{\omega} \in \Omega$ *belonging to support of* P *belongs also to the closure of* $\Re$. *Indeed, if support of* $\hat{\mathsf{Q}}_n$ *is a closed neighborhood of* $\hat{\omega}$ *with radius* $\frac{1}{n}$ *and* $\{\alpha_n\}_{n\leq 1}$*:* $\alpha_n > 0$, $\alpha_n \uparrow 1$ *while* $n \to \infty$, *then* $\mathsf{Q}_n \triangleq \alpha_n \hat{\mathsf{Q}}_n + (1 - \alpha_n)\mathsf{Q}$ *belong to* $\Re$ *for any* $n \geq 1$ *and converges weakly to Dirak measure of* $\hat{\omega}$ *(i.e.* $\mathsf{E}^{\mathsf{Q}_n} g(\omega) \to g(\hat{\omega})$ *for any bounded continuous* g*).*

Proof of Theorem 13. Suppose there exists solution of problem (2), namely, triplet $\left(\mathsf{Q}^*, \gamma_1^{*N}, \overline{V}_0\right)$. It is worth to mention, that non-redundance of initial $(1, S)$-market (with respect to P) guarantees, that it will be non-redundant with respect to the worst-case measure Q^* (might be proved similarly to Corollary 2). Hence, for any $t \in N_1$, the support of regular martingale conditional probability $\mathsf{Q}^*\left[\cdot|\mathcal{F}_t^S\right]$ consists of at least $d+1$ elements. If the supports above consist of $d+1$ each, we have assertion of the Theorem.

Else, let us consider discreet function of sets $\hat{\mathsf{Q}}$ specified by:

(1) equality $\hat{\mathsf{Q}}_0(A) \triangleq \mathsf{Q}^*(A)$ for any $A \in \mathcal{F}_0^S$;

(2) set of variables $\{\triangle \hat{x}_{t,j}, \hat{p}_{t,j}\}_{t\in N_1, 1\leq j\leq d+1}$, where (i) for any $t \in N_1$ elements of the set $\{\triangle \hat{x}_{t,j}\}_{t\in N_1, 1\leq j\leq d+1}$ are an affine-independent $\mathcal{F}_{t-1}^S$-measurable random variables belonging to the support of regular conditional probability $\mathsf{Q}^*\left[\cdot|\mathcal{F}_{t-1}^S\right]$ (if specified by Ω) such, that for any $t \in N_1$ the following system (with respect to d-dimensional γ and 1-dimensional z) is incompatible:

$$(\triangle \hat{x}_{t,j}, \gamma) \geq -z, -z > 0 \quad 1 \leq j \leq d+1; \tag{110}$$

(such a set exists because: (a) support of $\mathsf{Q}^*\left[\cdot|\mathcal{F}_t^S\right]$ has a full basis; (b) if the system for $j \in \{1, ..., d\}$ is compatible, then Lemma 1 guarantees existence of such $\triangle \hat{x}_{t,d+1}$, that $(\triangle \hat{x}_{t,d+1}, \gamma) < 0$).

(ii) for any $t \in N_1$ elements of the set $\{\hat{p}_{t,j}\}_{t\in N_1, 1\leq j\leq d+1}$ are defined as the only solution for the problem:

$$\sum_{j=1}^{d+1} \hat{p}_{t,j} \triangle \hat{x}_{t,j} = 0, \quad \sum_{i\geq 1} p_{t,i} = 1. \tag{111}$$

Note, that from the theory for systems of linear equations it is known [14], that system (111) has non-negative solution, if and only if (110) is incompatible.

Thus, $p_{t,i} \triangleq \hat{\mathsf{Q}}\left(\triangle S_t = \triangle x_{t,i} | \mathcal{F}^S_{t-1}\right)$, $t \in N_1$, $j = 1, ..., d+1$. Note, that constructed measure $\hat{\mathsf{Q}}$ belongs to close of $\mathfrak{R}_N$ in topology of weak convergence of probability measures (see Remark 11).

As $\{\triangle \hat{x}_{t,j}\}_{t \in N_1, 1 \le j \le d+1}$ is a subset of the support for regular conditional worst-case probability $\mathsf{Q}^* \left[\cdot | \mathcal{F}^S_{t-1}\right]$, $t \in N_1$, so $(\gamma^*_t, \triangle \hat{x}_{t,j}) = \ln \overline{V}_t(S_0, ..., S_{t-1}, S_{t-1} + \triangle \hat{x}_{t,j})$, $j = 1, ..., d+1$.

Now, according to Remark 10 triplet $\left(\hat{\mathsf{Q}}, \gamma^{*N}_1, \overline{V}_0\right)$ is a solution for problem (2) in non-redundant $(1, S)$-market, where regular conditional probabilities $\hat{\mathsf{Q}}\left(\cdot | \mathcal{F}^S_{n-1}\right)$, $t \in N_1$, are discreet and their supports consist of $d+1$ affine-independent predictable variables.

4.5. Proof of Theorems 14 and 15. The assertions follow from Theorems 10–13.

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МАТЕМАТИЧЕСКАЯ МОДЕЛЬ ЦЕНООБРАЗОВАНИЯ ДЛЯ ЕВРОПЕЙСКОГО ОПЦИОНА НА НЕПОЛНОМ РЫНКЕ БЕЗ ТРАНЗАКЦИОННЫХ ИЗДЕРЖЕК (ДИСКРЕТНОЕ ВРЕМЯ). ЧАСТЬ I.

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В статье построена модель ценообразования для европейского опциона на многомерном неполном рынке без транзакционных издержек с дискретным временем.С начала рассмотрена вспомогательная задача по нахождению верхнего гарантированного значения ожидаемого значения риска, экспоненциально зависящего от дефицита капитала. Верхнее гарантированное значение представляет собой минимаксное значение ожидаемого риска. Первой берется верхняя грань по множеству эквивалентных вероятностных мер, а затем – нижняя грань по множеству самофинансируемых портфелей. В статье найдены условия существования портфеля, на котором достигается нижняя грань. Этот результат позволил построить обобщение опционального разложения функции выплаты опциона. Затем получены условия существования вероятностной меры, доставляющей максимум ожидаемому значению риска. Эта мера оказалась мартингальной и дискретной, но в общем случае она не принадлежит множеству эквивалентных вероятностных мер. Наконец, показано, как полученные результаты для вспомогательной задачи позволяют получить явные формулы для цены европейского опциона на неполном рынке без транзакционных издержек. Во второй части статьи приведены примеры моделей ценообразования европейского опциона на рынках с одним рисковым активом: конечного и с компактным носителем базовой вероятностной меры.
Ключевые слова: европейский опцион, хеджирование, минимаксный портфель, неполный рынок, опциональное разложение, представление, функция риска.

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