MATHEMATICAL MODEL OF EUROPEAN OPTION PRICING IN INCOMPLETE MARKET WITHOUT TRANSACTION COSTS (DISCRETE TIME). PART II.

Oleg Zverev[1], Vladimir Khametov[2], Elena Shelemekh[1]

zv-oleg@yandex.ru, khametovvm@mail.ru, letis@mail.ru
[1]*Central Economic Mathematical Institute of RAS*
[2]*National Research University Higher School of Economics*

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This is the second part of the paper. Here general model of the first part is implemented to design pricing models for special cases of one-dimensional incomplete final market and compact (1; S)-market.
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Introduction.

This is second part of the paper. Here general results of the first part are implemented to design pricing models for special cases of European option in one-dimensional incomplete final market and compact $(1, S)$-market (both with discreet time and without transaction costs). The results are stated in sections 5 and 6. Particularly, in section 5 we construct superhedging and minimax portfolios for incomplete final $(1, S)$-market, specifies by relation (1). Here recurrent relation (5) from Theorem 1 of [3] is used to state, that the upper guaranteed value is a Markov random function (Theorem 16), and to find explicit formulas for one-step transition probabilities of random sequence (1) with respect to the worst-case measure Q^*. Example of calculation for European option in incomplete one-dimensional compact $(1, S)$-market is presented in section 6.

§5. Minimax hedging portfolio of a European option in a finite incomplete $(1, S)$-market

The aim of this section is to construct the minimax hedging portfolio of a European option in a finite $(1, S)$-market. Throughout this section we assume that, with respect to a basic measure P, the returns of the risky assets are represented by a sequence of independent and identically distributed random variables with finitely many values.

5.1. In this subsection, we introduce a finite $(1, S)$-market.

Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in N_0}, \mathsf{P}\right)$ be a stochastic basis. We assume that $\left\{S_t, \mathcal{F}_t^S\right\}_{t\in N_0}$ is a one-dimensional sequence of random variables defined on $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in N_0}, \mathsf{P}\right)$ such that, for each $t \in N_0$, this sequence admits the representation

$$S_t = S_{t-1}\left(1+\rho_t\right), \quad S_t|_{t=0} = S_0 > 0, \tag{1}$$

where S_0 is nonrandom and $\{\rho_t\}_{t\in N_1}$ is a sequence of random variables. From the economic standpoint, ρ_t is the return of the risky asset at the moment $t \in N_1$. Suppose that $0 < S_0 \le c_5$. Let us also assume that the sequence $\{\rho_t\}_{t\in N_1}$ satisfies the following conditions.

<u>*Conditions*</u> (ρ):

(1) $\{\rho_t\}_{t\in N_1}$ is a sequence of independent and identically distributed random variables;

(2) for any $t \in N_1$, the random variable ρ_t takes values in the set $\Gamma \triangleq \{a_1, \ldots, a_l\}$ with probabilities $p_1, \ldots, p_l$, where $p_i = \mathsf{P}\left(\rho_t = a_i\right)$, $i = \overline{1, l}$; moreover,

(a) $2 \le l < \infty$;

(b) $\inf\limits_{1\le i\le l} a_i > -1$, $\sup\limits_{1\le i\le l} a_i < \infty$;

(c) there is no $i \in \{2, \ldots, l\}$ such that $a_i = 0$;

(d) there are $j, k \in \{1, \ldots, l\}$ such that $a_j < 0$, $a_k > 0$.

It follows from (ρ) that

(1) ρ_t is an l-valued random variable with multinomial distribution such that it admits the representation

$$\rho_t = \sum_{i=1}^{l} a_i 1_{\{\rho_t = a_i\}},$$

where $1_{\{\rho_t = a_i\}} = \begin{cases} 1, & \rho_t = a_i \\ 0, & \rho_t \ne a_i \end{cases}$;

(2) without loss of generality, we can assume that $-1 < a_1 < a_2 < \cdots < a_l < \infty$;

(3) for any $t \in N_0$ random variable $S_t > 0$;

(4) the sequence $\{S_t\}_{t\in N_0}$ defined by recurrent relation (1) is a homogeneous Markov chain with respect to the basic measure P.

5.2. We need the following notation and remarks.

Clearly, the introduced $(1, S)$-market without transaction costs is incomplete for $l \ge 3$.

Let random sequence $\{\rho_t\}_{t\in N_1}$ satisfy conditions (ρ) and $\Re^d_{N,l}$ be a set of probability measures on trajectories of this random sequence. Obviously $\Re^d_{N,l} \ne \varnothing$. For any P and $\mathsf{Q} \in \Re^d_{N,l}$ we define $p_i \triangleq \mathsf{P}\left(\rho_t = a_i\right)$, $q_i \triangleq \mathsf{Q}\left(\rho_t = a_i\right)$.

<u>*Conditions*</u> $\left(\Re^d_{N,l}\right)$:

(1) with respect to any measure $\mathsf{Q} \in \Re^d_{N,l}$ random variables $\{\rho_t\}_{t\in N_1}$ are independent and identically distributed;

(2) if $\mathsf{P}, \mathsf{Q} \in \Re^d_{N,l}$, then for any $i \in \{1, ..., l\}$ we have $0 < p_i < 1$, $0 < q_i < 1$, $\sum\limits_{i=1}^{l} p_i = \sum\limits_{i=1}^{l} q_i = 1$.

If P, $\mathsf{Q} \in \Re^d_{N,l}$, then for any $i \in \{1, \ldots, l\}$, $0 < p_i < 1$, $0 < q_i < 1$, and $\sum\limits_{i=1}^{l} p_i = \sum\limits_{i=1}^{l} q_i = 1$. In this case, the Radon–Nikodym derivative of the probability measure Q with respect to the probability measure P allows the representation

$$\frac{d\mathsf{Q}}{d\,\mathsf{P}}\left(\rho_1, \ldots, \rho_N\right) = \prod_{j=1}^{l} \left(\frac{q_j}{p_j}\right)^{\sum\limits_{t=1}^{N} 1_{\left\{\rho_t = a_j\right\}}}$$

(see [6]).

Note that in this case $\Re^d_{N,l}$ is a convex weakly compact set.

Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^1$ be a bounded borelean function denoted by $\varphi(x)$. Assume that the contingent claim has the form $\varphi(S_N) = \varphi(x)\,|_{x=S_N}$.

In this section, we consider the problem of constructing the minimax hedging portfolio for the European contingent claim $\varphi(S_N)$ with a time horizon N in the introduced incomplete finite $(1, S)$-market.

Obviously, Theorems 10–15 of [3] hold in this case. Therefore, there is a probability measure Q^*_l with respect to which the $(1, S)$-market in consideration is the worst-case complete one. However, results of [3] does not give any explicit formula for the one-step transition probability for the sequence $(S_t, \mathcal{F}_t)_{t\in N_0}$ with respect to the measure Q^*. Hence, to find the explicit form of these probabilities and to construct the minimax hedging portfolio, we consider recurrent relation (5) from [3] (with our assumptions taken into account).

5.3. In this subsection, we consider recurrent relation (8) from [3] with the above remarks taken into account.

By $\overline{V}^l_t$ we denote the $\mathcal{F}^S_t$-measurable random variable

$$\overline{V}^l_t \triangleq \inf_{\gamma^N_{t+1} \in D^N_{t+1}} \sup_{\mathsf{Q} \in \Re^d_{N,l}} \mathsf{E}^{\mathsf{Q}} \left[\exp\left\{ \varphi(S_N) - \sum_{i=t+1}^{N} \gamma_i \Delta S_i \right\} |\mathcal{F}^S_t \right]. \qquad (2)$$

A reasoning similar to that used to prove Theorem 1 in [3] shows that, in this case, the sequence $\left\{\overline{V}_t^l, \mathcal{F}_t^S\right\}_{t\in N_1}$ satisfies the recurrent relation

$$\begin{cases} \overline{V}_{t-1}^l = \inf\limits_{\gamma\in D_t} \sup\limits_{\mathsf{Q}\in\Re_{N,l}^d} \mathsf{E}^{\mathsf{Q}}\left[\overline{V}_t^l e^{-\gamma S_{t-1}\rho_t} | \mathcal{F}_{t-1}^S\right] \\ \overline{V}_t^l|_{t=N} = e^{\varphi(S_N)}. \end{cases} \tag{3}$$

The following theorem is the main result of this section.

Theorem 16 *Suppose that conditions* (ρ) *and* $\left(\Re_{N,l}^d\right)$ *hold and the sequence* $\left\{S_t, \mathcal{F}_t^S\right\}_{t\in N_0}$ *satisfies recurrent relation (1). Let* $\varphi(x)$ *be a bounded borelean function. We also assume that* $\overline{V}_t^l$ *defined by (2) satisfies recurrent relation (3).*

Then the following statements are true.

(1) $D_t = \mathbb{R}^1$ *for any* $t \in N_0$.

(2) There is a borelean function $\overline{V}_t^l(x)$ *on* $N_0 \times \mathbb{R}^+$ *ranging in* $\mathbb{R}^+$ *such that for any* $t \in N_0$, *we have* $\overline{V}_t^l = \overline{V}_t^l(x)|_{x=S_t}$ P*-a.s.*

Moreover, for any $t \in N_1$ *and* $x \in \mathbb{R}^+$, *the function* $\overline{V}_t^l(x)$ *satisfies the recurrent relation*

$$\begin{cases} \overline{V}_{t-1}^l(x) = \inf\limits_{\gamma\in\mathbb{R}^1} \sup\limits_{\substack{0<q_i<1, \quad i=\overline{1,l}, \\ \sum_{i=1}^l q_i = 1}} \left(\sum\limits_{i=1}^l \overline{V}_t^l\left(x\left(1+a_i\right)\right) e^{-\gamma x a_i} q_i\right) \\ \overline{V}_t^l|_{t=N} = e^{\varphi(x)} \end{cases}. \tag{4}$$

Proof of Theorem 16. (1) We prove that for any $t \in N_1$, the set $D_t = \mathbb{R}^1$. It is sufficient to prove that for any $t \in N_1$ and $\gamma \in \mathbb{R}^1$,

$$\sup_{\mathsf{Q}\in\Re_{N,l}^d} \mathsf{E}^{\mathsf{Q}}\left[\overline{V}_t^l e^{-\gamma S_{t-1}\rho_t} | \mathcal{F}_{t-1}^S\right] < \infty \quad \mathsf{P}\text{-a.s.} \tag{5}$$

Note that (1) if $\sup\limits_{x\in\mathbb{R}^1} |\varphi(x)| \le c_6$, where $c_6 > 0$ is a constant, then for any $t \in N_0$,

$$0 < \overline{V}_t^l \le e^{c_6}. \tag{6}$$

The reasoning we used to prove inequality (52) in [3] (see the proof of Theorem 2 in [3]) is just as good for inequality (6). Therefore, the proof of (6) is omitted.

(2) Representation (1) implies that for any $t \in N_0$, the random variable S_t admits the representation

$$S_t = S_0 \prod_{i=1}^{t} (1 + \rho_i) . \tag{7}$$

If $\mathcal{F}_t^{\rho} \triangleq \sigma \{S_0, \rho_1, \dots, \rho_t\}$, then (7) means that for any $t \in N_1$, $\mathcal{F}_t^S = \mathcal{F}_t^{\rho}$. Further, it follows from (7) and conditions (ρ) that for any $t \in N_0$, there is a constant $c_7 > 0$ such that

$$0 < S_t \le c_5 (1 + a_l)^t \le c_7. \tag{8}$$

Consider $\mathsf{E}^{\mathsf{Q}} \left[\overline{V}_t^l e^{-(\gamma, \Delta S_t)} | \mathcal{F}_{t-1}^S \right]$, where $t \in N_1$, $\mathsf{Q} \in \Re_{N,l}^d$, and $\gamma \in D_t$ are arbitrary. Using Dynkin–Evstigneev's lemma [2], we obtain

$$0 \le \mathsf{E}^{\mathsf{Q}} \left[\overline{V}_t^l e^{-\gamma S_{t-1} \rho_t} | \mathcal{F}_{t-1}^S \right] = \mathsf{E}^{\mathsf{Q}} \left[\overline{V}_t^l e^{-\gamma x \rho_t} | \mathcal{F}_{t-1}^S \right] \Big|_{x = S_{t-1}} . \tag{9}$$

The fact that $\{\rho_t\}_{t \in N_1}$ is a family of mutually independent random variables with respect to any measure $\mathsf{Q} \in \Re_{N,l}^d$, inequalities (6) and (9), and conditions (ρ) imply that for any $t \in N_1$ and $x, \gamma \in \mathbb{R}^1$, the inequalities hold

$$\mathsf{E}^{\mathsf{Q}} \left[\overline{V}_t^l e^{-\gamma x \rho_t} | \mathcal{F}_{t-1}^S \right] \le e^{c_6} \mathsf{E}^{\mathsf{Q}} \left[e^{-\gamma x \rho_t} | \mathcal{F}_{t-1}^S \right] = \tag{10}$$

$$= e^{c_6} \mathsf{E}^{\mathsf{Q}} e^{-\gamma x \rho_t} = e^{c_6} \sum_{i=1}^{l} e^{-\gamma x a_i} q_i \le e^{c_6} l e^{|\gamma||x| a_l} < \infty.$$

Formula (10) and conditions (ρ) imply inequality (5) for any $\gamma \in \mathbb{R}^1$. Therefore, for any $t \in N_1$, we obtain $D_t = \mathbb{R}^1$.

Now we are going to prove that $\overline{V}_t^l$ is a Markov random function. This means that there exists a borelean function $\overline{V}_t^l(x)$ on $N_0 \times \mathbb{R}^+$ ranging in $\mathbb{R}^+$ and such that for each $t \in N_1$, there the representation $\overline{V}_t^l = \overline{V}_t^l(x)|_{x=S_t}$ holds. To prove this, we shall need a few auxiliary remarks. Since $\overline{V}_t^l$ is an $\mathcal{F}_t^S$-measurable function, Borel's theorem yields that for each $t \in N_0$, there exists a borelean function $\widetilde{V}_t(x_0, \dots, x_t)$ defined on $(\mathbb{R}^+)^{t+1}$ and ranging in $\mathbb{R}^+$, where $x_i \in \mathbb{R}^+$, $i = \overline{0, t}$, such that

$$\overline{V}_t^l = \widetilde{V}_t (x_0, \dots, x_t) \Big|_{x_i = S_i \ i = \overline{0,t}} .$$

Hence, it follows from (1), Dynkin–Evstigneev's lemma [2], and conditions (ρ) that for any $t \in N_1$, $\mathsf{Q} \in \Re_{N,l}^d$ and $\gamma \in D_t$, the equalities hold

$$\mathsf{E}^{\mathsf{Q}} \left[\overline{V}_t^l e^{-\gamma S_{t-1} \rho_t} | \mathcal{F}_{t-1}^S \right] = \mathsf{E}^{\mathsf{Q}} \left[\widetilde{V}_t (S_0, \dots, S_{t-1}, S_t) e^{-\gamma S_{t-1} \rho_t} | \mathcal{F}_{t-1}^S \right] =$$

$$= \mathsf{E}^{\mathsf{Q}}\left[\widetilde{V}_t\left(S_0,\ldots,S_{t-1},S_{t-1}\left(1+\rho_t\right)\right)e^{-\gamma S_{t-1}\rho_t}|\mathcal{F}^S_{t-1}\right]=$$

$$= \mathsf{E}^{\mathsf{Q}}\left[\widetilde{V}_t\left(x_0,\ldots,x_{t-1},x_{t-1}\left(1+\rho_t\right)\right)e^{-\gamma x_{t-1}\rho_t}|\mathcal{F}^S_{t-1}\right]\Big|_{x_i=S_i \quad i=\overline{0,t-1}}=$$

$$= \mathsf{E}^{\mathsf{Q}}\left[\widetilde{V}_t\left(x_0,\ldots,x_{t-1},x_{t-1}\left(1+\rho_t\right)\right)e^{-\gamma x_{t-1}\rho_t}\right]\Big|_{x_i=S_i \quad i=\overline{0,t-1}}=$$

$$= \sum_{i=1}^{l}\widetilde{V}_t\left(x_0,\ldots,x_{t-1},x_{t-1}\left(1+a_i\right)\right)e^{-\gamma x_{t-1}a_i}q_i\Bigg|_{x_i=S_i \quad i=\overline{0,t-1}}=$$

$$= \sum_{i=1}^{l}\widetilde{V}_t\left(S_0,\ldots,S_{t-1},S_{t-1}\left(1+a_i\right)\right)e^{-\gamma S_{t-1}a_i}q_i. \tag{11}$$

Formula (11) yields that for any $t\in N_1$ and $\gamma\in D_t$,

$$\sup_{\mathsf{Q}\in\Re^d_{N,l}}\mathsf{E}^{\mathsf{Q}}\left[\overline{V}^l_t e^{-\gamma S_{t-1}\rho_t}|\mathcal{F}^S_{t-1}\right]= \tag{12}$$

$$=\sup_{\left\{\begin{array}{c}0<q_i<1,\quad i=\overline{1,l},\\ \sum\limits_{i=1}^{l}q_i=1\end{array}\right\}}\sum_{i=1}^{l}\widetilde{V}_t\left(S_0,\ldots,S_{t-1},S_{t-1}\left(1+a_i\right)\right)e^{-\gamma S_{t-1}a_i}q_i.$$

Taking into account the above remarks and (12), we can rewrite (3) as

$$\widetilde{V}_t\left(S_0,\ldots,S_{t-1}\right)=$$

$$=\inf_{\gamma\in\mathbb{R}^1}\sup_{\left\{\begin{array}{c}0<q_i<1,\quad i=\overline{1,l},\\ \sum\limits_{i=1}^{l}q_i=1\end{array}\right\}}\sum_{i=1}^{l}\Big[\widetilde{V}_t\left(S_0,\ldots,S_{t-1},S_{t-1}\left(1+a_i\right)\right)\times$$
$$\times e^{-\gamma S_{t-1}a_i}q_i\Big].$$

Let us now prove that $\overline{V}^l_t$ is a Markov random function. At first, we will prove that for any $t\in N_0$, there is a borelean function $\overline{V}^l_t(x)$ ranging in $\mathbb{R}^+$ denoted by $\overline{V}^l_t(x)$ and such that $\overline{V}^l_t(S_t)\triangleq\overline{V}^l_t(x)\,|_{x=S_t}$ satisfies recurrent relation (13). We proceed by the backward induction. Since $\overline{V}^l_t|_{t=N}=e^{\varphi(S_N)}$, our assertion is true for $t=N$.

Suppose that $\overline{V}_t^l = \overline{V}_t^l(S_t)$. We must prove that $\overline{V}_{t-1}^l = \overline{V}_{t-1}^l(S_{t-1})$. Recurrent relation (13) yields that

$$\overline{V}_{t-1}^l = \inf_{\gamma\in\mathbb{R}^1} \sup_{\left\{\begin{array}{c} 0<q_i<1, \quad i=\overline{1,l}, \\ \sum\limits_{i=1}^{l} q_i = 1 \end{array}\right\}} \sum_{i=1}^{l} \overline{V}_t^l\left(S_{t-1}\left(1+a_i\right)\right) e^{-\gamma S_{t-1} a_i} q_i. \tag{13}$$

Since the right-hand side of (13) is measurable with respect to the σ-algebra $\sigma\left\{S_{t-1}\right\}$ generated by the random variable S_{t-1}, the left-hand side of (13) is also measurable with respect to the σ-algebra $\sigma\left\{S_{t-1}\right\}$. Therefore, $\overline{V}_t^l$ is a Markov random function. Thus, recurrent relation (13) takes the form (4). This completes the proof of the theorem.

5.4. In this subsection, we prove that the inner supremum and the outer infimum in recurrent relation (4) are attained. We also prove the existence of the unique martingale measure in the considered $(1,S)$-market.

The main result of this subsection is the following theorem.

Theorem 17 *Let the assumptions of Theorem 16 be satisfied. Then the following assertions are true.*

(1) For any $t\in N_1$ and $x\in\mathbb{R}^+$, the borelean function $\ln\overline{V}_t^l(x)$ satisfies the recurrent relation

$$\left\{\begin{array}{l} \ln\overline{V}_{t-1}^l(x) = \inf\limits_{\gamma\in\mathbb{R}^1}\max\limits_{1\le i\le l}\left[\ln\overline{V}_t^l\left(x\left(1+a_i\right)\right) - \gamma x a_i\right] \\ \ln\overline{V}_t^l(x)\left.\right|_{t=N} = \varphi(x). \end{array}\right. \tag{14}$$

(2) There are reflections $i^*: N_1\times\mathbb{R}^+\to\Gamma$; $j^*: N_1\times\mathbb{R}^+\to\Gamma$ and $\gamma^*: N_1\times\mathbb{R}^+\to\mathbb{R}^1$ respectively denoted by $i_t^*(x)$, $j_t^*(x)$ and $\gamma_t^*(x)$ such that

$$\inf_{\gamma\in\mathbb{R}^1}\max_{1\le i\le l}\left[\ln\overline{V}_t^l\left(x\left(1+a_i\right)\right)-\gamma x a_i\right] = \ln\overline{V}_t^l\left(x\left(1+a_{i_t^*(x)}\right)\right) - \gamma_t^*(x)\, x a_{i_t^*(x)} =$$

$$= \ln\overline{V}_t^l\left(x\left(1+a_{j_t^*(x)}\right)\right) - \gamma_t^*(x)\, x a_{j_t^*(x)}, \tag{15}$$

moreover, for any $t\in N_1$ and $x\in\mathbb{R}^+$:

(a) $a_{i_t^*(x)}<0$, $a_{j_t^*(x)}>0$,

(b) $\gamma_t^*(x)$ can be calculated by the formula

$$\gamma_t^*(x) = \frac{1}{x\left(\left|a_{i_t^*(x)}\right| + a_{j_t^*(x)}\right)} \ln\frac{\overline{V}_t^l\left(x\left(1+a_{j_t^*(x)}\right)\right)}{\overline{V}_t^l\left(x\left(1+a_{i_t^*(x)}\right)\right)}. \tag{16}$$

(3) Formula (14) can be rewritten as

$$\begin{cases} \ln \overline{V}^l_{t-1}(x) = (1 - q^*_t(x)) \ln \overline{V}^l_t \left(x \left(1 + a_{i^*_t(x)}\right)\right) + q^*_t(x) \ln \overline{V}^l_t \left(x \left(1 + a_{j^*_t(x)}\right)\right) \\ \ln \overline{V}^l_t(x) \mid_{t=N} = \varphi(x), \end{cases} \tag{17}$$

where

$$q^*_t(x) = \frac{\left|a_{i^*_t(x)}\right|}{\left|a_{i^*_t(x)}\right| + a_{j^*_t(x)}}. \tag{18}$$

(4) There is a unique probability measure Q^*_l with respect to which: (i) the Markov random function $\left\{\ln \overline{V}^l_t(S_t), \mathcal{F}^S_t\right\}_{t\in N_0}$ satisfies recurrent relation (17), (ii) the sequence $\left(S_t, \mathcal{F}^S_t\right)_{t\in N_0}$ satisfying recurrent relation (1) is a homogeneous Markov chain. Moreover, for each $t \in N_1$, the random variable ρ_t takes two values: $a_{i^*_t(S_{t-1})}$ or $a_{j^*_t(S_{t-1})}$ with the conditional probabilities

$$\mathsf{Q}^*\left(\rho_t = a_{i^*_t(S_{t-1})} | S_{t-1}\right) = 1 - q^*_t(S_{t-1}) \tag{19}$$

$$\left(\mathsf{Q}^*\left(\rho_t = a_{j^*_t(S_{t-1})} | S_{t-1}\right) = q^*_t(S_{t-1})\right),$$

where $q^*_t(S_{t-1}) \triangleq q^*_t(x)\mid_{x=S_{t-1}}$ and $q^*_t(x)$ is defined by (18), (iii) the measure Q^* is a unique martingale measure, i.e., for each $t \in N_1$,

$$\mathsf{E}^{\mathsf{Q}^*}\left(\rho_t | \mathcal{F}^S_{t-1}\right) = 0. \tag{20}$$

P R O O F O F T H E O R E M 17. (1) First, we note that for any $t \in N_1$ and $x \in \mathbb{R}^+$, we have

$$\sup_{\left\{\begin{array}{c} 0 < q_i < 1, \quad i = \overline{1,l}, \\ \sum_{i=1}^{l} q_i = 1 \end{array}\right\}} \sum_{i=1}^{l} \overline{V}^l_t(x(1+a_i)) e^{-\gamma x a_i} q_i =$$

$$= \max_{\left\{\begin{array}{c} 0 \le q_i \le 1, \quad i = \overline{1,l}, \\ \sum_{i=1}^{l} q_i = 1 \end{array}\right\}} \sum_{i=1}^{l} \overline{V}^l_t(x(1+a_i)) e^{-\gamma x a_i} q_i.$$

It is clear that

$$\max_{\left\{\begin{array}{c} 0 \le q_i \le 1, \quad i = \overline{1,l}, \\ \sum_{i=1}^{l} q_i = 1 \end{array}\right\}} \sum_{i=1}^{l} \overline{V}^l_t(x(1+a_i)) e^{-\gamma x a_i} q_i = \tag{21}$$

$$= \max_{1\le i\le l}\overline{V}^{l}_{t}\left(x\left(1+a_i\right)\right)e^{-\gamma x a_i}.$$

Therefore, combining recurrent relation (4) with (21), we obtain

$$\overline{V}^{l}_{t-1}\left(x\right) = \inf_{\gamma\in\mathbb{R}^1}\max_{1\le i\le l}\overline{V}^{l}_{t}\left(x\left(1+a_i\right)\right)e^{-\gamma x a_i}. \tag{22}$$

Since for any $t \in N_1$ and $x \in \mathbb{R}^+$, the function $\overline{V}^{l}_{t}\left(x\right) > 0$, formula (22) implies that $\ln \overline{V}^{l}_{t}\left(x\right)$ satisfies (14).

(2) By definition

$$\psi\left(t,x,\gamma\right) \triangleq \max_{1\le i\le l}\left[\ln \overline{V}^{l}_{t}\left(x\left(1+a_i\right)\right) - \gamma x a_i\right].$$

For any (t,x), the function $\psi\left(t,x,\gamma\right)$ is the upper envelope [5] of the set of functions $\left\{\ln \overline{V}^{l}_{t}\left(x\left(1+a_i\right)\right) - \gamma x a_i\right\}_{i=\overline{1,l}}$ treated as functions of $\gamma \in \mathbb{R}^1$. It is easy to verify that for any (t,x), the function $\psi\left(t,x,\gamma\right)$ is a continuous, piecewise linear, convex, bounded from below function of $\gamma \in \mathbb{R}^1$. Moreover,

$$\psi\left(t,x,\gamma\right) \underset{|\gamma|\to\infty}{\rightarrow} \infty.$$

Therefore, there exists a borelean function $\gamma^* : N_0 \times \mathbb{R}^+ \to \mathbb{R}^1$ denoted by $\gamma^*_t\left(x\right)$ such that

$$\inf_{\gamma\in\mathbb{R}^1}\psi\left(t,x,\gamma\right) = \psi\left(t,x,\gamma^*_t\left(x\right)\right).$$

Let us find the explicit formula for $\gamma^*_t\left(x\right)$. It follows from the properties of the function $\psi\left(t,x,\gamma\right)$ and conditions (ρ) that for each (t,x), there are $i^*_t\left(x\right)$ and $j^*_t\left(x\right)$ ranging in Γ and such that:

(a) $a_{i^*_t(x)} < 0$ and

$$\psi\left(t,x,\gamma^*_t\left(x\right)\right) = \ln \overline{V}^{l}_{t}\left(x\left(1+a_{i^*_t(x)}\right)\right) - \gamma^*_t\left(x\right) x a_{i^*_t(x)}, \tag{23}$$

(b) $a_{j^*_t(x)} > 0$ and

$$\psi\left(t,x,\gamma^*_t\left(x\right)\right) = \ln \overline{V}^{l}_{t}\left(x\left(1+a_{j^*_t(x)}\right)\right) - \gamma^*_t\left(x\right) x a_{j^*_t(x)}. \tag{24}$$

It is obvious that for each (t,x), we have $i^*_t\left(x\right) < j^*_t\left(x\right)$. Formulas (23) and (24) yield that $\gamma^*_t\left(x\right)$ satisfies the equality

$$\ln \overline{V}^{l}_{t}\left(x\left(1+a_{i^*_t(x)}\right)\right) - \gamma^*_t\left(x\right) x a_{i^*_t(x)} = \tag{25}$$

$$= \ln \overline{V}^{l}_{t}\left(x\left(1+a_{j^*_t(x)}\right)\right) - \gamma^*_t\left(x\right) x a_{j^*_t(x)}.$$

Solving (25) for $\gamma_t^*(x)$, we obtain (16).

(3) Applying (16), (24), and some elementary transformations to recurrent relation (14), we obtain (17) and (18).

(4) Formulas (17) and (18) imply that there exists a probability measure Q^* such that: (i) $\{S_t\}_{t\in N_0}$ is an inhomogeneous Markov chain, (ii) for any $t \in N_1$, the random variable ρ_t takes values $a_{i_t^*(S_{t-1})}$ and $a_{j_t^*(S_{t-1})}$ with the conditional probabilities $\mathsf{Q}^*\left(\rho_t = a_{i_t^*(S_{t-1})}|S_{t-1}\right) = 1 - q_t^*(S_{t-1})$ and $\mathsf{Q}^*\left(\rho_t = a_{j_t^*(S_{t-1})}|S_{t-1}\right) = q_t^*(S_{t-1})$, respectively, where $q_t^*(S_{t-1}) = q_t^*(x)|_{x=S_{t-1}}$ and $q_t^*(x)$ is defined by (18). Hence, we have (20). Formula (20) implies that the measure Q^* is a unique martingale measure. This completes the proof of the theorem.

5.5. In this subsection, we prove that the measure Q^* constructed in Subsection 5.3 is the worst-case measure.

Theorem 18 *Let the assumptions of Theorem 17 be satisfied. Then the probability measure* Q^* *is the worst-case one.*

PROOF OF THEOREM 18. Assume the opposite, i.e., assume that Q^* is not the worst-case measure. It follows from (13) that there is $t \in N_0$ such that

$$\begin{aligned} 1 &= \inf_{\gamma\in\mathbb{R}^1} \sup_{\mathsf{Q}\in\Re^d_{N,l}} \mathsf{E}^{\mathsf{Q}}\left[\frac{\overline{V}^l_t}{\overline{V}^l_{t-1}} e^{-\gamma\Delta S_t} |\mathcal{F}^S_{t-1}\right] > \\ &> \inf_{\gamma\in\mathbb{R}^1} \mathsf{E}^{\mathsf{Q}^*}\left[\exp\left\{\Delta\ln\overline{V}^l_t - \gamma\Delta S_t\right\}|\mathcal{F}^S_{t-1}\right]. \end{aligned} \tag{26}$$

We have already proved (see the second assertion of Theorem 17) that there is an $\mathcal{F}_{t-1}$-measurable random variable $\gamma_t^* \triangleq \gamma_t^*(x)|_{x=S_{t-1}}$ such that

$$\begin{aligned} &\inf_{\gamma\in\mathbb{R}^1} \mathsf{E}^{\mathsf{Q}^*}\left[\exp\left\{\Delta\ln\overline{V}^l_t - \gamma\Delta S_t\right\}|\mathcal{F}^S_{t-1}\right] = \\ &= \mathsf{E}^{\mathsf{Q}^*}\left[\exp\left\{\Delta\ln\overline{V}^l_t - \gamma^*_t\Delta S_t\right\}|\mathcal{F}^S_{t-1}\right]. \end{aligned}$$

Inequality (26) and the last equality yield

$$0 > \ln \mathsf{E}^{\mathsf{Q}^*}\left[\exp\left\{\Delta\ln\overline{V}^l_t - \gamma^*_t\Delta S_t\right\}|\mathcal{F}^S_{t-1}\right]. \tag{27}$$

Applying Jensen's inequality to (27), we have

$$0 > \mathsf{E}^{\mathsf{Q}^*}\left[\Delta\ln\overline{V}^l_t - \gamma^*_t\Delta S_t|\mathcal{F}^S_{t-1}\right]. \tag{28}$$

On the other hand, (20) imply the martingale property of the sequence $\{S_t, \mathcal{F}_t\}_{t\in N_0}$ with respect to the measure Q^*. Therefore, using (17) and (20), we obtain

$$0 = \mathsf{E}^{\mathsf{Q}^*}\left[\Delta \ln \overline{V}^l_t | \mathcal{F}^S_{t-1}\right] = \mathsf{E}^{\mathsf{Q}^*}\left[\Delta \ln \overline{V}^l_t - \gamma^*_t \Delta S_t | \mathcal{F}^S_{t-1}\right] \tag{29}$$

Comparing (28) with (29) we obtain a contradiction. Thus, Q^* is the worst-case measure. This completes the proof of the theorem.

5.6. In this subsection, we prove that the contingent claim admits an S-representation with respect to the measure Q^*. The main result of this subsection is the following theorem.

Theorem 19 *Suppose that the assumptions of Theorem 18 are satisfied. Then any bounded contingent claim $\varphi(S_N)$ admits an S-representation with respect to the martingale measure Q^*, i.e.,*

$$\varphi(S_N) = \mathsf{E}^{\mathsf{Q}^*}\left[\varphi(S_N)|\mathcal{F}_0\right] + \sum_{i=1}^{N} \gamma^*_i(S_{i-1}) S_{i-1}\rho_i, \tag{30}$$

where $\gamma^*_i(S_{i-1})$ is defined by (16).

PROOF OF THEOREM 19. Note that it follows from Theorems 17 and 18 that we can rewrite recurrent relation (14) as

$$1 = \mathsf{E}^{\mathsf{Q}^*}\left[\exp\left\{\Delta \ln \overline{V}^l_t - \gamma^*_t \Delta S_t\right\} | \mathcal{F}^S_{t-1}\right]. \tag{31}$$

Let us prove that the random sequence $\left\{\ln \overline{V}^l_t, \mathcal{F}_t\right\}_{t\in N_0}$ satisfies the recurrent relation

$$\begin{cases} \Delta \ln \overline{V}^l_t = \gamma^*_t \Delta S_t, \\ \ln \overline{V}^l_t|_{t=0} = \ln \overline{V}^l_0, \quad \ln \overline{V}^l_t|_{t=N} = \varphi(S_N) \end{cases} \tag{32}$$

with respect to measure Q^*.

Indeed, on the one hand, it follows from (31) and Jensen's inequality that for any $t \in N_1$,

$$\begin{aligned} 0 &= \ln \mathsf{E}^{\mathsf{Q}^*}\left[\exp\left\{\Delta \ln \overline{V}^l_t - \gamma^*_t \Delta S_t\right\} | \mathcal{F}^S_{t-1}\right] \geq \\ &\geq \mathsf{E}^{\mathsf{Q}^*}\left[\Delta \ln \overline{V}^l_t - \gamma^*_t \Delta S_t | \mathcal{F}^S_{t-1}\right] \end{aligned} \tag{33}$$

On the other hand, in the proof of Theorem 18, equality (29) was shown to be true. It is clear that inequality (33) becomes the equality if and only if the random variable $\left(\Delta \ln \overline{V}_t^l - \gamma_t^* \Delta S_t\right)$ is $\mathcal{F}_{t-1}^S$-measurable. Therefore, we obtain recurrent relation (32). It is obvious that $\ln \overline{V}_t^l|_{t=0} = \ln \overline{V}_0^l$ and $\ln \overline{V}_t^l|_{t=N} = \varphi(S_N)$. Since Q^* is a martingale measure, (32) yields S-representation (30) and the equality

$$\ln \overline{V}_0^l = \mathsf{E}^{\mathsf{Q}^*} \left[\varphi(S_N) | \mathcal{F}_0\right]$$

with respect to Q^*. This completes the proof of the theorem.

5.7. In this subsection, we construct the minimax hedging portfolio of an European option in a finite $(1, S)$-market.

Theorem 20 *Let $\varphi(S_N)$ be a bounded contingent claim. Suppose that the assumptions of Theorem 19 are satisfied. Then the finite incomplete $(1, S)$-market described by recurrent relation (1) is the worst-case complete market, i.e., there exists a measure Q^* and a minimax hedging self-financing portfolio $\pi^* = (\beta_t^*, \gamma_t^*)_{t \in N_0}$ such that:*

(1) for any $t \in N_1$, the predictable sequence $(\gamma_t^*)_{t \in N_0}$ is defined by (16), and γ_0^* can be chosen to be zero;

(2) the predictable sequence $(\beta_t^*)_{t \in N_0}$ is defined by the recurrent relation

$$\beta_t^* = \beta_{t-1}^* - S_{t-1} \Delta \gamma_t^*, \quad \beta_t^*|_{t=0} = \beta_0^*, \tag{34}$$

and β_0^* can be chosen equal to $\ln \overline{V}_0^l(S_0)$;

(3) the capital $X_t^{\pi^*}$ of the portfolio π^* at any moment $t \in N_0$ admits the representations

$$X_t^{\pi^*} = \beta_t^* + \gamma_t^* S_t \tag{35}$$

$$X_t^{\pi^*} = X_0^{\pi^*} + \sum_{i=1}^{t} \gamma_i^* S_{i-1} \rho_i, \tag{36}$$

where $X_0^{\pi^*} = \ln \overline{V}_0^l(S_0)$;

$$X_t^{\pi^*} = \ln \overline{V}_t^l(S_t),$$

where $\ln \overline{V}_t^l(S_t)$ satisfies recurrent relation (17) and $X_N^{\pi^*} = \varphi(S_N)$.

Proof of Theorem 20. Since the amount of the risky asset γ_t^* at any moment $t \in N_1$ is defined by (16), we can use the self-financing condition (18) from [3] to obtain recurrent relation (34) for the amount of the riskless asset β_t^*. Therefore, the capital $X_t^{\pi^*}$ of the portfolio $\pi^* = (\beta_t^*, \gamma_t^*)_{t \in N_0}$ allows representation (35) for any $t \in N_0$. Formulas (35) and (18) from [3] imply that $\Delta X_t^{\pi^*}$ admits the representation

Comparing (32) with (37), for any $t \in N_1$, we obtain

$$\Delta X_t^{\pi^*} = \Delta \ln \overline{V}_t^l. \qquad (38)$$

Let us choose $X_0^{\pi^*} = \ln \overline{V}_0^l$. We use (38) to show that for any $t \in N_0$, it is true that $X_t^{\pi^*} = \ln \overline{V}_t^l$. Therefore, $X_N^{\pi^*} = \ln \overline{V}_N^l = \varphi(S_N)$. Thus, the $(1, S)$-market described by recurrent relation (1) is the worst-case complete one, and the portfolio π^* described by recurrent relations (16), (17) and (34) is the minimax hedging portfolio with respect to the measure Q^*. This completes the proof of the theorem.

Remark 11 *(1) Condition* $(\rho_{c)})$ *can be omitted. This will lead to more sophisticated formulas for the initial price of the option and for the portfolio components.*

(2) Suppose that $q^* = \mathsf{Q}^*(\rho_t = a) = \frac{b}{|a|+b}$, $p^* = \mathsf{Q}^*(\rho_t = b) = \frac{|a|}{|a|+b}$, *where* $-1 < a < 0 < b < \infty$, *are unique. Following the same reasoning, it is easy to prove that such a binomial* $(1, S)$*-market is the worst-case one.*

§6. An example.

In this section, we consider an example of calculating the minimax hedging portfolio for a European option in a $(1, S)$-market under the assumption that the return of the risky asset is a sequence of independent and identically distributed random variables with respect to a basic measure P such that their probability distribution has a compact support.

6.1. Let $\{S_t, \mathcal{F}_t^S\}_{t \in N_0}$ be an adapted random sequence of prices on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in N_0}, \mathsf{P})$. Suppose that $\{S_t, \mathcal{F}_t^S\}_{t \in N_0}$ satisfies recurrent relation (1). Assume that the random sequence $\{\rho_t, \mathcal{F}_t^S\}_{t \in N_1}$ has the following properties with respect to the basic measure P:

(i) $\{\rho_t, \mathcal{F}_t^S\}_{t \in N_1}$ are independent and identically distributed random variables;

(ii) $[a, b]$, where $-1 < a < 0 < b < \infty$, is the support of the probability distribution of the random variable ρ_t.

In this case, for any $t \in N_0$, the random variable $S_t > 0$ P-a.s. It follows from the above assumptions that the sequence $\{S_t\}_{t \in N_0}$ is a homogeneous Markov sequence with respect to the measure P.

Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^1$ be a bounded borelean function denoted by $\varphi(x)$. Let $\varphi(S_N) = \varphi(x)|_{x = S_N}$ be a contingent claim.

By $\mathbb{M}[a, b]$ we denote the set of probability measures with support $[a, b]$. By definition $\mathbb{M}^N[a, b] \triangleq \underbrace{\mathbb{M}[a, b] \times \cdots \times \mathbb{M}[a, b]}_{N}$. It is well known (see [1]) that $\mathbb{M}[a,b]$ and $\mathbb{M}^N[a,b]$ are convex compact sets (in the topology of weak convergence $\sigma\left(\mathbb{M}^N, \left(\mathbb{M}^N\right)^*\right)$).

Let $\Re^c_N$ be a subset of $\Re_N$ such that: (1) the random variables $\{\rho_t\}_{t\geq 1}$ are independent and identically distributed with respect to any measure $\mathsf{Q}_N \in \Re^c_N$; (2) any measure $\mathsf{Q}_N \in \Re^c_N$ is absolutely continuous with respect to the Lebesque measure.

Let us summarize some properties of the set $\Re^c_N$:

(1) $\Re^c_N \neq \varnothing$ and $\Re^c_N \subset \mathbb{M}^N[a,b]$;

(2) $\Re^c_N$ is a convex weakly compact set;

(3) the sequence defined by (1) is a homogeneous Markov sequence with respect to any measure $\mathsf{Q}_N \in \Re^c_N$.

It is easy to verify that the above-introduced $(1,S)$-market is incomplete.

6.2. In this subsection, we construct the minimax hedging portfolio of a European contingent claim with time horizon N in the above-described $(1,S)$-market.

Note that, in this case, all assumptions of Theorems 5, 11, 13–15 from [3] are satisfied. Therefore there exists: (1) the worst-case (martingale and discrete) probability measure Q^*_N such that the considered $(1,S)$-market is the worst-case complete one with respect to Q^*_N; (2) the minimax hedging portfolio π^*. Therefore, $\mathsf{Q}^*_N \in \left(\mathbb{M}^N[a,b] \setminus \Re^c_N\right) \cap \mathfrak{M}_N$ is a unique (in a sense of Remark 7 from [3]) martingale measure. Hence Q^*_N is an extreme point of this set. Thus, using Choquet's theorem [4, ?], we can determine the form of the distribution of the random variable ρ_t (with respect to the measure Q^*_N) and construct the minimax hedging portfolio π^*.

6.3. In this subsection, we prove that $\{a,b\}^N$ is the support of the measure Q^*_N.

Since $\mathsf{Q}^*_N \in \left(\mathbb{M}^N[a,b] \setminus \Re^c_N\right) \cap \mathfrak{M}_N$, this is a product measure, i.e., $\mathsf{Q}^*_N = \underbrace{\mathsf{Q}^*_1 \times \cdots \times \mathsf{Q}^*_1}_{N}$, where $\mathsf{Q}^*_1 \in \mathbb{M}[a,b]$. Since Q^*_N is a martingale measure, we have

$$\mathsf{E}^{\mathsf{Q}^*_N}\rho_t = 0.$$

Therefore, zero is the barycenter [4] of the measure Q^*_1. It is obvious that the measure Q^*_1 satisfies the assumptions of Choquet's theorem [4, ?]. Hence, the support of the measure Q^*_1 is a subset of the set of extreme points of $[a,b]$. Thus, we see that the random variable ρ_1 takes the values a and b with respect to the measure Q^*_1 with probabilities $q^* = \mathsf{Q}^*_1(\rho_1 = a)$ and $p^* = 1 - q^*$, respectively. Since zero is the barycenter of the measure Q^*_1, we see that $q^* = \frac{b}{b+|a|}$. Thus, we have proved that $\{a,b\}^N$ is a support of the measure Q^*_N.

6.4. We denote

$$\overline{V}_t^c \triangleq \operatorname*{ess\,inf}_{\gamma_{t+1}^N \in D_{t+1}^N} \operatorname*{ess\,sup}_{\mathsf{Q}_N \in \Re_N^c} \mathsf{E}^{\mathsf{Q}_N} \left[\exp\left\{ \varphi(S_N) - \sum_{i=t+1}^N \gamma_i S_{i-1} \rho_i \right\} | \mathcal{F}_t^S \right]. \qquad (39)$$

Since Q_N^* is the worst-case martingale probability measure, we can rewrite (39) as

$$\overline{V}_t^c \triangleq \inf_{\gamma_{t+1}^N \in D_{t+1}^N} \mathsf{E}^{\mathsf{Q}_N^*} \left[\exp\left\{ \varphi(S_N) - \sum_{i=t+1}^N \gamma_i S_{i-1} \rho_i \right\} | \mathcal{F}_t^S \right].$$

Then Theorems 4 and 6 from [3] imply that $\left(\overline{V}_t^c, \mathcal{F}_t^S\right)_{t \in N_0}$ satisfies the recurrent relation

$$\begin{cases} \overline{V}_t^c = \inf\limits_{\gamma \in D_t} \mathsf{E}^{\mathsf{Q}_N^*} \left[\overline{V}_{t+1}^c e^{-\gamma S_t \rho_{t+1}} | \mathcal{F}_t^S \right] \\ \overline{V}_t^c|_{t=N} = e^{\varphi(S_N)}. \end{cases} \qquad (40)$$

It follows from the results of Subsection 6.3 that for any $t \in N_0$, the set $\{a, b\}$ is a support of the conditional probability distribution $\mathsf{Q}_N^*(\cdot|\mathcal{F}_t^S)$. Therefore, a reasoning similar to those we used in Subsections 5.2 and 5.3 proves that for any $t \in N_0$,

(1) $D_t = \mathbb{R}^1$,

(2) there exists a borelean function $\overline{V}_t^c(x)$ such that

(a) $\overline{V}_t^c = \overline{V}_t^c(x)|_{x=S_t}$,

(b) $\overline{V}_t^c(x)$ satisfies the recurrent relation

$$\begin{cases} \overline{V}_{t-1}^c(x) = \inf\limits_{\gamma \in \mathbb{R}^1} \left[\overline{V}_t^c(x(1+a)) e^{\gamma x |a|} q^* + \overline{V}_t^c(x(1+b)) e^{-\gamma x b} p^* \right] \\ \overline{V}_t^c(x)|_{t=N} = e^{\varphi(x)}. \end{cases} \qquad (41)$$

Note that $\left[\overline{V}_t^c(x(1+a)) e^{\gamma x|a|} q^* + \overline{V}_t^c(x(1+b)) e^{-\gamma x b} p^*\right]$ is a strictly convex function of $\gamma \in \mathbb{R}^1$. Therefore, there exists a unique function $\gamma_t^*(x)$ defined on $N_1 \times \mathbb{R}^+$ and ranging in $\mathbb{R}^1$ such that

$$\inf_{\gamma \in \mathbb{R}^1} \left[\overline{V}_t^c(x(1+a)) e^{\gamma x|a|} q^* + \overline{V}_t^c(x(1+b)) e^{-\gamma x b} p^* \right] = \qquad (42)$$

$$= \overline{V}_t^c(x(1+a)) e^{\gamma_t^*(x) x |a|} q^* + \overline{V}_t^c(x(1+b)) e^{-\gamma_t^*(x) x b} p^*.$$

Formula (42) yields the equality

$$\gamma_t^*(x) = \frac{1}{x(b+|a|)} \ln \frac{\overline{V}_t^c(x(1+b))}{\overline{V}_t^c(x(1+a))}. \qquad (43)$$

Combining (41) with (42) and (43) and applying elementary transformations to the result, we obtain

$$\begin{cases} \overline{V}^c_{t-1}(x) = \left(\overline{V}^c_t(x(1+a))\right)^{q^*} \left(\overline{V}^c_t(x(1+b))\right)^{p^*} \\ \overline{V}^c_t(x)\,|_{t=N} = e^{\varphi(x)}. \end{cases} \tag{44}$$

This yields the recurrent relation

$$\begin{cases} \ln \overline{V}^c_{t-1}(x) = q^* \ln \overline{V}^c_t(x(1+a)) + p^* \ln \overline{V}^c_t(x(1+b)) \\ \ln \overline{V}^c_t(x)\,|_{t=N} = \varphi(x). \end{cases} \tag{45}$$

It is easy to verify that the solution of recurrent relation (45) allows the representation

$$\ln \overline{V}^c_t(x) = \sum_{i=0}^{N-t} \varphi\left(x(1+a)^i (1+b)^{N-t-i}\right) C^i_{N-t} (q^*)^i (p^*)^{N-t-i}. \tag{46}$$

This coincides with the well-known formula (3) from [6] (see p. 744).

Formulas (43) and (46) give the explicit representation of the amount of the risky asset $\gamma^*_t = \gamma^*_t(x)|_{x=S_{t-1}}$ at any moment $t \in N_1$.

Self-financing condition (18) from [3] implies that the amount of the riskless asset β^*_t at any moment $t \in N_0$ satisfies the recurrent relation

$$\begin{cases} \beta^*_t = \beta^*_{t-1} - S_{t-1}\Delta\gamma^*_t, \\ \beta^*_t|_{t=0} = \beta^*_0. \end{cases}$$

Without loss of generality, we can assume that β^*_0 is equal to $\ln \overline{V}^c_0$ and $\gamma^*_0 = 0$. Thus, we have just constructed the self-financing portfolio $\pi^* = (\beta^*_t, \gamma^*_t)_{t \in N_0}$. It follows from Theorem 15 of [3] that for any $t \in N_0$, the capital $X^{\pi^*}_t$ of the portfolio $\pi^* \in SF$ admits the representation

$$X^{\pi^*}_t = \ln \overline{V}^c_t,$$

where $\ln \overline{V}^c_t = \ln \overline{V}^c_t(x)\,|_{x=S_t}$. Moreover,

(1) $X^{\pi^*}_t|_{t=N} = \varphi(S_N)$,

(2) the initial capital $X^{\pi^*}_0 = \ln \overline{V}^c_0$,

(3) $X^{\pi^*}_t = X^{\pi^*}_0 + \sum\limits_{i=1}^{t} \gamma^*_i S_{i-1}\rho_i$.

Theorems 14–15 from [3] imply that the considered $(1, S)$-market is the worst-case complete market and π^* is the minimax hedging portfolio.

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МАТЕМАТИЧЕСКАЯ МОДЕЛЬ ЦЕНООБРАЗОВАНИЯ ДЛЯ ЕВРОПЕЙСКОГО ОПЦИОНА НА НЕПОЛНОМ РЫНКЕ БЕЗ ТРАНЗАКЦИОННЫХ ИЗДЕРЖЕК (ДИСКРЕТНОЕ ВРЕМЯ). ЧАСТЬ II.

Зверев О.В.[1], Хаметов В.М.[2], Шелемех Е.А.[1]

zv-oleg@yandex.ru, khametovvm@mail.ru, letis@mail.ru
sbulgakov@hse.ru, vmkhametov@hse.ru
[1]*Центральный экономико-математический институт РАН*
[2]*Национальный исследовательский университет «Высшая школа экономики»*

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Это вторая часть статьи. Здесь модель общего характера, построенная в первой части, использована для построения моделей ценообразования в частных случаях одномерного неполного конечного рынка с компактным носителем.
Ключевые слова: европейский опцион, хеджирование, минимаксный портфель, неполный рынок, компактный рынок.

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